Is Matter an emergent property of Space-Time?

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The first, fully non linear, mean field theory of relativistic gravitation has been developed in 2004. The theory makes the striking prediction that averaging or coarse graining a gravitational field changes the matter content of space-time. A review of the general theory is presented, together with new calculations that highlight this effect on black holes of the Reisner-Nordström (RN) family. Explicit expressions for the equation of state and for the energy and charge densities of the apparent matter surrounding the coarse grained black holes are given. In particular, the Schwarzschild black hole, which is a vacuum solution of Einstein field equations, is shown to appear after coarse graining as surrounded by an apparent matter whose equation of state strongly resembles the equation of state commonly postulated for cosmological dark energy. Thermodynamical aspects are also investigated. Taken together, these new results suggest that matter and charge may be properties of the space-time which only emerge after a certain coarse graining has been performed.

1 Introduction

Developing a mean field theory for general relativity has long been the subject of active research ([9, 13, 10, 11, 20, 1, 2]). This problem is of undeniable theoretical interest, but it is also of real practical importance because finite precision effects in astrophysical observations of relativistic objects and in observational cosmology can only be modeled properly through a mean field theory of relativistic gravitation ([8]).

The last three years have witnessed the construction of the first general mean field theory for Einstein gravitation ([5, 6]). The theory makes the striking prediction that averaging or coarse graining a gravitational field changes the matter content of space-time. In particular, the net 'large scale' effect of the averaged upon, 'small scale' gravitational degrees is to contribute an apparent 'large scale' matter which self consistently generates the coarse grained gravitational field. This matter is charged if the gravitational field is coupled to an electromagnetic field. These remarkable predictions raise the intriguing possibility that matter may simply be an emergent property of space time.

We first introduce the general theory mean field theory. We then present coarse grainings of both the Schwarschild and the extreme Reisner-Nordström black holes. We derive explicit expressions for the equation of state and for the energy and charge densities of the apparent matter surrounding the coarse grained black holes. In particular, the Schwarzschild black hole, which is a vacuum solution of Einstein field equations, is shown to appear after coarse graining as surrounded by an apparent matter whose equation of state strongly resembles the equation of state commonly postulated for cosmological dark energy. We also investigate thermodynamical aspects and prove that the envisaged coarse graining transforms the extreme RN black hole, which has a vanishing temperature, into a black hole of non vanishing temperature.

2 A mean field theory for general relativity

Let $\mathcal{M}$ be a fixed 'base' manifold and let Ω be an arbitrary probability space. Let $g(\omega)$ be an ω-dependent Lorentzian metric defined on $\mathcal{M}$; let also $A(\omega)$ be an ω-dependent 4-potential, with associated current $j(\omega)$. Each triplet $\mathcal{S}(\omega) = (\mathcal{M}, g(\omega), A(\omega))$ represents a physical space-time; we denote by Σ the collection of all these space-times. Each member $\mathcal{S}(\omega)$ of Σ is naturally equipped with the Levi-Civita connection $\Gamma(\omega)$ of the metric $g(\omega)$ [19]; to $\mathcal{S}(\omega)$ is also associated a stress-energy tensor $\mathcal{T}(\omega)$ related to the metric $g(\omega)$, the connection $\Gamma(\omega)$ and $A(\omega)$ through Einstein equation; we write $\mathcal{T}(\omega) = \mathcal{T}_{(A(\omega),g(\omega))} + \mathcal{T}_m(\omega)$, where $\mathcal{T}_{(A(\omega),g(\omega))}$ represents the stress-energy tensor generated by $A(\omega)$ in $g(\omega)$ and $\mathcal{T}_m(\omega)$ represents the stress-energy of other matter fields.

It has been shown in ([5]) that the collection Σ of space-times can be used to define a single, mean or coarse grained space-time $\bar{\mathcal{S}} = (\mathcal{M}, \bar{g}, \bar{A})$ where the metric $\bar{g}$ and the potential $\bar{A}$ are the respective averages of the metrics $g(\omega)$ and of the potentials $A(\omega)$ over ω; thus, for all points P of $\mathcal{M}$, $\bar{g}(P) = \langle g(P, \omega)\rangle$ and $\bar{A}(P) = \langle A(P, \omega)\rangle$, where the brackets on the right-hand side indicate an average over the probability space Ω.

The connection of the mean space-time $\bar{\mathcal{S}}$ is simply the Levi-Civita connection associated to the metric $\bar{g}$ and will be conveniently called the mean or coarse grained connection. Since the relations linking the coordinate basis components $g_{\mu\nu}$ of an arbitrary metric g to the Christoffel symbols $\Gamma^{\alpha}_{\mu\nu}$ of its Levi-Civita connection are non-linear, the Christoffel symbols of the mean connection are *not* identical to the averages of the Christoffel symbols associated to the various space-times $\mathcal{S}(\omega)$.

The metric $\bar{g}$ and its Levi-Civita connection $\bar{\Gamma}$ define an Einstein tensor $\bar{\mathcal{E}}$ for the coarse grained space-time $\bar{\mathcal{S}}$. This tensor defines, via Einstein equation, the stress-energy tensor $\bar{\mathcal{T}}$ for $\bar{\mathcal{S}}$. Because Einstein equation is non linear in both the metric and the electromagnetic potential, $\bar{\mathcal{T}}^{\alpha\beta}$ is generally different from $\left\langle \mathcal{T}_m^{\alpha\beta}(\omega) \right\rangle + \mathcal{T}_{\bar{A},\bar{g}}$. The additional, generally non vanishing tensor field $\Delta\mathcal{T} = \bar{\mathcal{T}} - \langle \mathcal{T}_m(\omega) \rangle - \mathcal{T}_{\bar{A},\bar{g}}$, can be interpreted as the stress-energy tensor of an 'apparent matter'. This apparent matter simply describes the cumulative effects of the averaged upon (small scale) fluctuations of the gravitational and electromagnetic fields on the (large scale) behaviour of the coarse grained gravitational field. In particular, the vanishing of $\mathcal{T}(\omega)$ for all ω does not necessarily imply the vanishing of $\bar{\mathcal{T}}$. The mean stress-energy tensor $\bar{\mathcal{T}}$ can therefore be non vanishing in regions where the unaveraged stress-energy tensor actually vanishes.

The Maxwell equation relating the electromagnetic potential to the electromagnetic current also couples the electromagnetic and the gravitational field non linearly; the mean current $\bar{j}$ associated to $\bar{A}$ in $\bar{g}$ does not therefore coincide with the average $< j(\omega) >$. In particular, a region of space-time where $j(\omega)$ vanishes for all ω is generally endowed with a non vanishing mean current $\bar{j}$.

Let us finally mention that the averaging scheme just presented is the only one which ensures that the motions in a mean field can actually be interpreted, at least locally, as the averages of 'real' unaveraged motions. This very important point is fully developed in ([6]).

3 Example 1: Coarse graining a Schwarzschild black hole

3.1 Determination of the coarse grained space-time

We consider a collection of space-times ([7, 4]) which describes a single Schwarzschild black hole observed with finite precision measurements of the three spatial Kerr-Schild coordinates. More precisely, this collection is defined as an ensemble of space-times $\mathcal{S}(\omega)$, each member of the ensemble being equipped with the metric $g(t, \mathbf{r}, \boldsymbol{\omega})$ given by:

$$ds_\omega^2 = dt^2 - d\mathbf{r}^2 - \frac{2M}{|\mathbf{r} - \omega|}\left(dt - \frac{(\mathbf{r} - \omega).d\mathbf{r}}{|\mathbf{r} - \omega|}\right)^2 \tag{1}$$

The parameter M represents the mass of the black hole and $\mathbf{r}$ stands for the set of three 'spatial' coordinates x, y, z. The set Ω of possible values for $\boldsymbol{\omega}$ is taken to be the Euclidean 3-ball of radius a: $\Omega = \{\boldsymbol{\omega} \in \mathbb{R}^3; \omega^2 \leqslant a^2\}$. The probability measure on Ω is defined by its uniform density $p(\boldsymbol{\omega}) = 3/(4\pi a^3)$ with respect to the Lebesgue measure $d^3\omega$ and all 3-D scalar products and norms in (1) are Euclidean. The electromagnetic field vanishes identically in all space-times $\mathcal{S}(\boldsymbol{\omega})$. The coarse graining is caracterized by the dimensionless parameter $x = a/M$.

The exact expression of the mean metric $\bar{g}$ corresponding to this collection can be obtained for every $a < r$. The mean metric expressed in Kerr-Schild coordinates is given by:

$$\begin{aligned}\left\langle ds^2\right\rangle = &\left(1 - \frac{2M}{r}\right)dt^2 - \left(\frac{2M}{r} - \frac{6a^2M}{5r^3}\right)\left(\frac{\mathbf{r}}{r}\cdot d\mathbf{r}\right)^2 - \left(1 + \frac{2a^2M}{5r^3}\right)d\mathbf{r}^2 \\ &+ \left[-\frac{3M}{2r} - \frac{3Mr}{2a^2} + \frac{3M}{4a^3r^2}\left(a^2 - r^2\right)^2 \ln\left(\frac{r+a}{r-a}\right)\right]\frac{\mathbf{r}}{r}\cdot d\mathbf{r}dt.\end{aligned} \tag{2}$$

One can construct a new set of coordinates $(\tau, \rho, \theta, \phi)$ (conveniently called Schwarzschild coordinates) which makes the static and spherically symmetric character of the mean space-time apparent. The metric then takes the form:

$$\left\langle ds^2\right\rangle = F(\rho)d\tau^2 - G(\rho)d\rho^2 - \rho^2 d\Gamma^2 \tag{3}$$

where $d\Gamma^2$ stands for the usual volume element on the unit sphere S^2. The exact expressions of both $F(\rho)$ and $G(\rho)$ need not be reproduced here. We mention however that $\rho_H = 2M\sqrt{1 + x^2/20}$, which corresponds to $r = 2M$, is a first order pole of G and a zero of F; it is also the first encountered singularity of G when coming form infinity in ρ-space. The coarse grained space-time is thus a black hole of horizon radius ρ_H.

3.2 Energetics of the coarse grained black hole

Stress-energy tensor The exact expressions of the Schwarzschild components of the mean stress-energy tensor are too complicated to warrant reproduction here. We just present the approximate expressions of these components, which are valid when $a \ll r$:

$$8\pi \bar{\mathcal{T}}^0_0 = \varepsilon = -\frac{6a^2M^2}{5\rho^6}; \qquad 8\pi \bar{\mathcal{T}}^1_1 = -p_1 = -\frac{6a^2M^2}{5\rho^6};$$
$$8\pi \bar{\mathcal{T}}^2_2 = -p_2 = \frac{12a^2M^2}{5\rho^6}; \qquad 8\pi \bar{\mathcal{T}}^3_3 = -p_3 = \frac{12a^2M^2}{5\rho^6}. \tag{4}$$

This shows that the coarse graining procedure endows the original vacuum space-time with a non vanishing stress-energy tensor $\bar{\mathcal{T}}$. This tensor describes how the averaged upon (small scale) degrees of freedom of the Schwarzschild gravitational field can be viewed as an apparent matter which acts as the effective 'source' of the coarse grained (large scale) field. The apparent matter is characterized by a negative energy density and an anisotropic pressure tensor. Note that all energy conditions (*i.e.* the weak, strong and dominant energy conditions ([19])) are violated by the mean stress energy tensor $\bar{T}$. Finally, by taking the trace of Einstein's equation, the scalar curvature $\bar{\mathcal{R}}$ of the mean space-time outside the horizon can be obtained directly from the exact components of $\bar{\mathcal{T}}$; one finds, at second order in a/ρ:

$$\bar{\mathcal{R}} = -8\pi \bar{\mathcal{T}}^\mu_\mu = -\frac{12a^2M^2}{5\rho^6}. \tag{5}$$

The coarse graining thus endows the space-time with a negative scalar curvature. This inevitably evokes the recent observations ([18]) of a positive, non-vanishing cosmological constant Λ, which also endows vacuum regions of space-time with a negative scalar curvature ([15]) $\mathcal{R}_\Lambda = -4\Lambda$. The similarity and differences between the coarse grained space-time constructed here and space-times of cosmological interest are further explored in [7].

Mass and temperature of the coarse grained black hole We have just seen that the coarse graining changes the repartition of energy in space-time. A standard asymptotic analysis at spatial or null infinity shows that the coarse grained space-time described by (3) is asymptotically flat and that its mass is identical to M, the mass of the unaveraged Schwarzschild black hole. This result is true for all values of a (including those superior to $2M$).

The temperature $T(x, M)$ of the coarse grained black hole can be obtained by studying the natural topology of the associated Euclidean space time ([19]). One obtains

$$T(x, M) = -\frac{1}{12\pi M}\,\frac{x^3}{-x\left(1 + \frac{x^2}{4}\right) + \left(1 - \frac{x^2}{4}\right)^2 \ln\left(\frac{1+x/2}{1-x/2}\right)}. \tag{6}$$

One finds that $T(x, M) \approx \dfrac{1}{8\pi M}\left(1 + \dfrac{x^2}{20}\right)$ at order two in x; this confirms that $T(0, M)$ coincides with the Hawking temperature $1/8\pi M$ ([19]) of the Schwarzschild black hole.

4 Example 2: Coarse graining of an extreme Reisner-Nordström black hole

Type of coarse graining Extreme black holes are thermodynamically particularly interesting because they have a vanishing temperature. One can thus wonder if a coarse graining similar to the one applied above to the Schwarzschild black hole would not transform the extreme black hole into a black hole of finite temperature. The answer is negative because the coarse grained space-time turns out not to be a black hole. There exist however a simple complex generalization of the above coarse graining which does transform the real extreme Reisner-Nordström black hole into a real black hole of non vanishing temperature. Note that complex space-times have been considered in a wide variety of context, which range from spinor and twistor theory [16] to black hole physics [3] and string theory [17]; the natural occurence of complex space-times in the present problem therefore comes as no surprise.

Consider a collection of complex space-times $\mathcal{S}(\omega)$ equipped with the metric

$$ds^2_\omega = dt^2 - d\mathbf{r}^2 - h_\omega(r)\left(dt - \frac{(\mathbf{r} - i\omega).d\mathbf{r}}{R(\mathbf{r}, \omega)}\right)^2, \tag{7}$$

with $R(\mathbf{r}, \omega) = \left(r^2 - \omega^2 - 2\,i\,\mathbf{r}.\omega\right)^{1/2}$, and $h_\omega(r) = 1 - \left(1 - \frac{M}{R(\mathbf{r},\omega)}\right)^2$. The four space-time coordinates t and $\mathbf{r}$ are still assumed real and the probability space Ω to which the parameter ω belongs is the same as in Section 3. The principal determination is retained in the definition of R, with cut along the positive imaginary axis. Note that the metric corresponding to $\omega = 0$ is the real extreme RN metric in Kerr-Schild coordinates. The only matter outside each black hole is an electromagnetic field of 4-potential

$$A_t(\omega) = -\frac{M}{R} \qquad A_x(\omega) = \frac{M}{R^2}\,h_\omega(r)\,x \tag{8}$$

$$A_y(\omega) = \frac{M}{R^2}\,h_\omega(r)\,y \qquad A_z(\omega) = \frac{M}{R^2}\,h_\omega(r)\,z. \tag{9}$$

The associated current $j(\omega)$ vanishes identically but the total charge of each space-time is $Q = M$.

One can obtain an explicit expression of the metric $\bar{g}$ of the complex coarse grained space-time $\bar{S}$ for all $r > a$. This metric is real and can be put into the manifestly static and spherically symmetric form (3). A standard analysis then shows that the coarse grained space-time is a real black hole; the horizon radius $\rho_H(x, M)$ is given at second order in x by $\rho_H(x, M) \simeq M\left(1 + \frac{x}{\sqrt{5}} - \frac{3x^2}{\sqrt{10}}\right)$.

Mass and charge repartition It is straightforward to evaluate the potential $\bar{A} =< A(\omega) >$ of $\bar{\mathcal{S}}$ and the corresponding stress-energy tensor $T_{\bar{A},\bar{g}}$. One can also evaluate the total stress-energy tensor $\bar{\mathcal{T}}$ of $\bar{\mathcal{S}}$. The difference $\Delta\bar{\mathcal{T}}$ yields the stress-energy tensor of the apparent matter describing the net large scale effect of the averaged upon, small scale degrees of freedom. One finds, at second order in a/ρ:

$$8\pi\,(\Delta\bar{\mathcal{T}})^0_0 = a^2\left(\frac{4M^3}{5\rho^7} - \frac{4M^4}{5\rho^8}\right); \qquad 8\pi\,(\Delta\bar{\mathcal{T}})^1_1 = a^2\left(\frac{4M^2}{5\rho^6} - \frac{4M^3}{5\rho^7}\right);$$
$$8\pi\,(\Delta\bar{\mathcal{T}})^2_2 = a^2\left(-\frac{8M^2}{5\rho^6} + \frac{24M^3}{5\rho^7} - \frac{2M^4}{\rho^8}\right); \qquad 8\pi\,(\Delta\bar{\mathcal{T}})^3_3 = 8\pi\,(\Delta\bar{\mathcal{T}})^2_2. \tag{10}$$

However, a direct calculation shows that the total mass is unaltered by the coarse graining.

The 4-current $\bar{j}$ in $\bar{\mathcal{S}}$ can be evaluated from $\bar{A}$ and $\bar{g}$. One finds that $\bar{\mathbf{j}} = 0$ but $\bar{j}^0 = \dfrac{a^2 M^3\,(2 - 3\,\rho/M)}{5\,\pi\,\rho^7}$ at second order in a/ρ. A direct computation shows that the total charge of $\bar{\mathcal{S}}$ is equal to $Q = M$, the charge of the original extreme black hole. Note however that the charge density $\bar{j}^0$ and Q are of opposite signs outside the horizon.

Temperature The temperature $T(x, M)$ of the coarse grained black hole can be obtained, as usual, by considering the Euclidean space-time associated to the region outside the horizon [19]. One obtains, at second order in x, that $T(x, M) \simeq \dfrac{x}{2\sqrt{5}\pi\,M} - \dfrac{x^2}{5\,\pi\,M}$. The coarse graining over complex space-time degrees of freedom has thus transformed the extreme real RN black hole of vanishing temperature into a real black hole of non vanishing temperature.

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