Chapter 1

Fat-tailed degree distributions generated by quenched disorder

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Some nongrowing networks can be investigated by using an urn model. We consider an urn model with a preference concept, i.e., "the rich get richer." After explaining the relationship between the urn model and a corresponding network model, we show numerically and analytically that quenched disorder states in the model play an important role to generate fat-tailed distributions; when each urn (node) has the same ability for obtaining balls (edges), the fat-tailed occupation (degree) distribution does not occur in the urn (network) model; when the ability of urns (nodes) are different from each other, the occupation (degree) distribution shows fat-tailed behavior.

1.1 Introduction

In recent years, complex networks have attracted a lot of attentions in statistical physics [1]. One of the important properties of complex networks is *scale-free* property; a network with the scale-free property is characterized by its fat-tailed degree distribution, $P(k) \sim k^{-\gamma}$, where γ is a characteristic exponent. It has been revealed that many real-world networks, such as Internet, World Wide Web, metabolic networks, social networks, have the scale-free property.

Numerous models have been proposed in order to explain the emergence of the scale-free property, and most of these models have two important concepts, i.e., the concepts of *preference* and *growth* [2]. The "preference" concept indicates the fact that "the rich get richer," and it seems reasonable for explaining various phenomena in real world. However, it is still an open question that the "growth" concept is suitable for all kind of real networks, so that some researchers have proposed nongrowing model for complex networks. For example, threshold models have succeeded in generating scale-free networks without growth. [7, 5, 12]. The threshold models implicitly have the preference concept, and an important point of the threshold models is existence of randomness; each node has a different parameter for connecting edges to the other nodes. Therefore, it is considered that the preference concept and the randomness concept might be important to generate the fat-tailed degree distribution.

The threshold models are not dynamical; we connect each pair of nodes by using a certain rule, and the network has no rewiring process. Nongrowing models with dynamical rewiring process have also been proposed, and it has been shown that the preference concept alone dose not give networks with the fat-tailed degree distribution [8, 14, 15]. In addition, it has been revealed that the randomness concept, i.e., the quenched disorder for the ability of obtaining edges, might be essential in order to generate the fat-tailed behavior [14, 15].

In the present paper, we investigate the importance of the preference and randomness (quenched disorder) concepts for generating fat-tailed behavior. In order to perform an analytical treatment, we propose an urn model. The relationship between a nongrowing network model and an urn model has been proposed in our previous paper [16], and then we here review the discussion. In addition, we present further numerical experiments and discussions to ensure our proposition, i.e., "*the preference concept and another additional concept would be important to generate fat-tailed behavior*."

1.2 Network Model and Urn Model

1.2.1 Nongrowing network model

First, we introduce a nongrowing network model with preferential rewiring processes.

(i) Set N nodes, and connect randomly these nodes with M edges; then, the initial network is an Erdös-Rényi network [10]. In addition, assign a fitness parameter β_i to each node i. Each fitness parameter represents an ability of the node for obtaining edges, and the value of the fitness parameters are chosen by using a fitness distribution $\phi(\beta)$.

(ii) Select an edge l_{ij} at random.

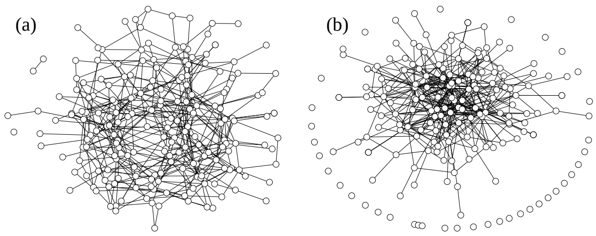

Figure 1.1: Examples of generated networks. Each network consists of $N = 100$ nodes, and the average degree is $\langle k \rangle = 4.0$ for both cases. We performed $100N$ rewiring processes. (a) The case with the fitness distribution $\phi(\beta) = \delta(\beta)$; (b) $\phi(\beta) = 1$ $(0 \leq \beta \leq 1)$.

(iii) Replace the edge l_{ij} by an edge l_{im}, where node m is chosen randomly with the following probability:

$$\Pi_m \propto (k_m + 1)^{\beta_m}, \tag{1.1}$$

where k_m is the degree of node m.

(iv) Repeat the procedures (ii) and (iii) until the system reaches an equilibrium state.

The examples of generated networks are shown in Fig. 1.1. When the fitness distribution is $\phi(\beta) = \delta(\beta)$, the probability Π_m is independent of each degree; then, a node to be attached to the rewiring edge is selected uniformly random. Hence, the generated network is an Erdös-Rényi network (Fig. 1.1(a)). On the other hands, in the case of the uniform fitness distribution, i.e., $\phi(\beta) = 1$ $(0 \leq \beta \leq 1)$, there are many isolated nodes, and an giant cluster exists (Fig. 1.1(b)). We note that the giant cluster has a fat-tailed degree distribution, as shown in the following sections.

1.2.2 Relationship between network models and urn models

In this subsection, we show a relationship between network models and urn models. Urn models have been used in order to explain various physical phenomena [4, 6, 9, 11, 18]. The urn model consists of urns and balls which are distributed among urns. There is a dynamics, so-called "ball-box" dynamics; we select a ball at random, and transfer the ball to the other urn selected by a certain rule[1].

Figure 1.2 shows a dynamics of the nongrowing network model and that of the corresponding urn model. Note that we can consider the degree of each node in

[1]There are a different dynamics, so-called "box-box" dynamics. In statistical physics, the ball-box dynamics corresponds to the classical system, and the box-box dynamics can be considered as the quantum one.

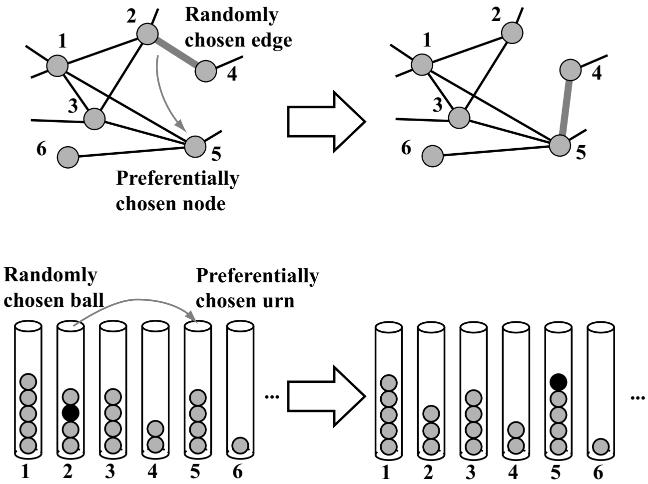

Figure 1.2: (upper) The nongworing network model with the preferential rewiring process. (bottom) A corresponding urn model. The number of edges connected to a node corresponds to that of balls within an urn. Therefore, in order to investigate the degree distribution of the network model, we can use the occupation distribution of the urn model.

the network model as the number of balls within each urn in the corresponding urn model. For instance, node 2 has four edges and lose an edge after the rewiring process; urn 2 has four balls and lose a ball after the ball-transfer process. Therefore, we can use the urn model in order to investigate the degree distribution of the network models; the occupation distribution of the urn model corresponds to the degree distribution of the network model.

The reason which we introduce the urn model is as follows; it is easy to analyze the urn models because the partition function is given by a factorized form [11]. Although we can construct the master equation for calculating the degree distribution using the transition rate Π_m (Eq. 1.1), it is difficult to solve the master equation for the above nongrowing network model because of the quenched disorder. On the other hands, occupation distributions of the urn models are relatively easily calculated by using the partition function, even in the case with quenched disorder; due to the existence of the quenched disorder, the analytical treatment for the urn model become a little complicated, and then it is needed to use the replica method [13] for the analysis. Hence, the urn models can be useful tools for analyzing network models.

1.2.3 Preferential urn model

In this subsection, we construct the urn model corresponding to the nongrowing network model described above. The urn model has the preference concept, i.e., "the rich get richer," so that we call the model *the preferential urn model.*

The urn model consists of M balls distributed among N urns[2]. We denote the density of the system as $\rho = M/N$, and the number of balls within urn i as n_i. Hence, $\sum_{i=1}^{N} n_i = M$.

We here define the energy of each urn as $E(n_i) = -\ln(n_i!)$; it is clear that each urn tends to get more and more balls because an urn is stable when the urn obtains a large number of balls. The unnormalized Boltzmann weight is written by $p_{n_i} = (n_i)^{\beta_i}$, where β_i is an inverse local temperature of urn i and corresponds to the fitness parameter in the nongrowing network model. Using the heat-bath rule, the transition rate $W_{n_i \to n_i+1}$ from the state n_i to $n_i + 1$ is written as $W_{n_i \to n_i+1} \propto \frac{p_{n_i+1}}{p_{n_i}} = (n_i + 1)^{\beta_i}$ [11]. Therefore, the preferential urn model has the same transition rate of the nongrowing network model.

The construction of the preferential urn model is as follows:

(i) Set N urns and M balls which is distributed at random. In addition, assign each inverse local temperature β_i to each urn i by using a distribution $\phi(\beta)$.

(ii) Choose a ball at random.

(iii) Transfer the chosen ball to an urn selected by using the transition rate $W_{n_i \to n_i+1}$.

(iv) Repeat the procedures (ii) and (iii) until the system reaches an equilibrium state.

1.3 Analytical and Numerical Results

1.3.1 Results for the case without quenched disorder

When there are no quenched disorder, i.e., the distribution of the inverse local temperatures, $\phi(\beta)$, has a delta-function form, the occupation distribution has an exponential decay [16, 17]. For example, we obtain a Poisson distribution analytically as the occupation distribution for the case $\phi(\beta) = \delta(\beta)$.

1.3.2 Analytical results for the uniform quenched disorder

For quenched disordered cases, the occupation distribution has a fat-tailed behavior. When the fitness distribution $\phi(\beta)$ is uniformly random, i.e., $\phi(\beta) = 1$ for $0 \le \beta \le 1$, we can obtain the analytical representation of the occupation distribution. Here, we only show the analytical results; for the details, see Ref. [17].

For the uniform quenched disordered case, the equilibrium occupation distribution is written as

$$P(k) = \int_0^1 \frac{(k!)^{\beta-1} z_{\mathrm{S}}^k}{\sum_{m=0}^{\infty} (m!)^{\beta-1} z_{\mathrm{S}}^m} d\beta, \tag{1.2}$$

[2]Note that one edge in the network model corresponds to two balls in the urn model.

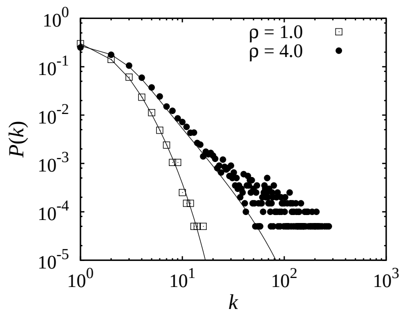

Figure 1.3: The occupation distributions in the case with $N = 1000$. The number of ball-transfer process is $500N$ times, and the distribution of the inverse local temperatures is the uniform one, i.e., $\phi(\beta) = 1$ $(0 \leq \beta \leq 1)$. The numerical results for the density $\rho = 1.0$ are shown by squares, and those for $\rho = 4.0$ by filled circles. The solid lines correspond to Eq. (1.2) for respective cases. The data are averaged over 20 different realizations.

where $z_{\rm S}$ is determined by solving the following equation:

$$\rho = \frac{z_{\rm S}\left[1 + \sum_{m=2}^{\infty}\left\{\frac{1}{\ln m!}\left(1 - \frac{1}{m!}\right)\right\} m z_{\rm S}^{m-1}\right]}{1 + z_{\rm S} + \sum_{m=2}^{\infty}\left\{\frac{1}{\ln m!}\left(1 - \frac{1}{m!}\right)\right\} z_{\rm S}^{m}}. \tag{1.3}$$

When $\rho \gg 1$, $z_{\rm S}$ is approximately equal to 1, so that the approximate form for the equilibrium occupation distribution in large k region is obtained as

$$P(k) \sim k^{-2}\frac{1}{(\ln k)^2}. \tag{1.4}$$

Therefore, the equilibrium occupation distribution follows a generalized power law with a squared inverse logarithmic correction. Note that the approximate behavior is similar to the Parato distribution.

The analytical and numerical results are shown in Fig. 1.3. The analytical results give good agreement with the numerical ones.

1.3.3 Numerical results for other quenched disordered cases

For the case in which the distribution of the inverse local temperatures, $\phi(\beta)$, is not uniform, we show only numerical results; analytical treatments for these cases are similar to that of the uniform quenched disordered case. Here, we consider the distribution $\phi(\beta)$ characterized by a parameter α; $\phi(\beta) = (\alpha+1)(1-\beta)^{\alpha}$ $(0 \leq \beta \leq 1)$. For $\alpha = 0$, we obtain the uniform distribution explained in the previous subsection. Figure 1.4 shows the occupation distributions obtained numerically. For the occupation distribution in the case with $\alpha = 1.0$, we obtain the power-law form with the exponents -2.98 ± 0.06; for $\alpha = 4.0$, the exponents

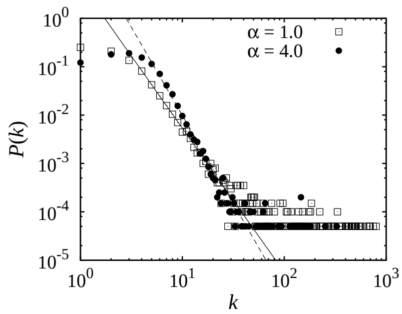

Figure 1.4: The occupation distribution in the case with $N = 1000$ and $500N$ ball-transfer processes. The distribution of the inverse local temperatures is given the form of $\phi(\beta) = (\alpha + 1)(1 - \beta)^{\alpha}$. Numerical results in the cases with $\alpha = 1.0$ and $\alpha = 4.0$ are shown. The density of balls is $\rho = 4.0$ for both cases. The numerical results for $\alpha = 1.0$ are shown by squares, and those for $\alpha = 4.0$ by filled circles. The solid line corresponds to $P(k) \sim k^{-2.98}$, and the dashed line indicates $P(k) \sim k^{-3.65}$.

-3.65 ± 0.04. Those numerical results show that the quenched disorder for the inverse local temperatures β is important to generate the fat-tailed behavior.

1.4 Concluding Remarks

In summary, we show the relationship between network models and urn models; the degree distribution of some nongrowing networks can be investigated by using urn models. Furthermore, using the analytical treatments and numerical experiments, we show that the fat-tailed behavior occurs due to the following two concepts; the preference concept and the quenched disorder concerning the ability for obtaining edges (balls). Therefore, we consider that the preference concept and another additional concept (enhancing the preference) are needed to generate the fat-tailed behavior. Furthermore, the growth concept may play the same role of randomness. Unified analysis of the mechanisms for the fat-tailed behavior should be needed in future works.

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