Chapter 1

Traffic flow in a spatial network model

Michael T. Gastner
Santa Fe Institute
1399 Hyde Park Road, Santa Fe, NM 87501
mgastner@santafe.edu

A quantity of practical importance in the design of an infrastructure network is the amount of traffic along different parts in the network. Traffic patterns primarily depend on the users' preference for short paths through the network and spatial constraints for building the necessary connections. Here we study the traffic distribution in a spatial network model which takes both of these considerations into account. Assuming users always travel along the shortest path available, the appropriate measure for traffic flow along the links is a generalization of the usual concept of "edge betweenness". We find that for networks with a minimal total maintenance cost, a small number of connections must handle a disproportionate amount of traffic. However, if users can travel more directly between different points in the network, the maximum traffic can be greatly reduced.

1.1 Introduction

In the last few years there has been a broad interdisciplinary effort in the analysis and modeling of networked systems such as the world wide web, the Internet, and biological, social, and infrastructure networks [1, 8, 19]. A network in its simplest form is a set of nodes or **vertices** joined together in pairs by lines or **edges**. In many examples, such as biochemical networks and citation networks, the vertices exist only in an abstract "network space" without a meaningful geometric interpretation. But in many other cases, such as the Internet, transportation or communication networks, vertices have well-defined positions in literal physical space, such as computers in the the Internet, airports in airline networks, or cell phones in wireless communication networks.

The spatial structure of these networks is of great importance for a better understanding of the networks' function and topology. Recently, several authors have proposed network models which depend explicitly on geometric space [2, 4, 9, 10, 12, 15, 16, 18, 20, 21, 22, 23, 24, 25, 26]. In all of these models, nearby vertices are more likely to be connected than vertices far apart. However, the importance of geometry manifests itself not only in the tendency to build short edges, but also in the traffic flow on the network: given a choice between different paths connecting two (not necessarily adjacent) vertices in the network, users will generally prefer the shortest path. With few exceptions [3, 5], the literature on spatial networks has rarely analyzed traffic patterns emerging from the various models. To address this issue, this paper takes a closer look at one particular model [13] and analyzes the distribution of traffic along the edges in the network.

In Sec. 1.2 we present the model whose objective is the creation of short paths between the vertices while keeping the total length of all edges small. In Sec. 1.3 we introduce a measure for the traffic flow based on the concept of "edge betweenness" and analyze the betweenness distribution for our network model. We conclude with a summary of our results in Sec. 1.4.

1.2 A model for optimal spatial networks

Suppose we are given the positions of n vertices, e.g. cities or airports, and we are charged with designing a network connecting these vertices together, e.g. with roads or flights. The efficiency of the network, as we will consider it here, depends on two factors. On the one hand, the smaller the sum of the lengths of all edges, the cheaper the network is to construct and maintain. On the other hand, the shorter the distances through the network between vertices, the faster the network can perform its intended function (e.g., transportation of passengers between nodes or distribution of mail or cargo). These two objectives generally oppose each other: a network with few and short connections will not provide many direct links between distant points and paths through the network will tend to be circuitous, while a network with a large number of direct links is usually expensive to build and operate. The optimal solution lies somewhere between these extremes.

Let us define l_{ij} to be the shortest geometric distance between two vertices i and j measured along the edges in the network. If there is no path between i and j, we formally set $l_{ij} = \infty$. Introducing the adjacency matrix $\mathbf{A}$ with elements $A_{ij} = 1$ if there is an edge between i and j and $A_{ij} = 0$ otherwise, we can write the total length of all edges as

$$T = \sum_{i<j} A_{ij} l_{ij}. \tag{1.1}$$

We assume this quantity to be proportional to the cost of maintaining the network. Clearly this assumption is only approximately correct: networked systems in the real world will have many factors affecting their maintenance costs that are not accounted for here. It is however the obvious first assumption to make and, as we will see, can provide us with good insight about network structure.

Besides maintenance, there is also a cost Z due to traveling through the network for the user. In a spirit similar to our assumption about maintenance costs, we will assume that the total travel cost is given by the sum of the distances between all vertex pairs. There is, however, one complicating factor. The travel costs are not necessarily proportional to *geometric* distances between vertices. In some cases, e. g. road networks, the quickest and cheapest route will indeed not be very different from the shortest route measured in kilometers. But in other networks travel costs depend more strongly on the *graph* distance, i. e. the number of legs in a journey. In an airline network, for instance, passengers often spend a lot of time waiting for connecting flights, so that they care more about the number of stopovers they have to make than about the physical distance traveled.

To model both cases we introduce two different expressions for the travel costs. For a road network, these costs are approximately

$$Z_1 = \sum_{i<j} l_{ij}, \tag{1.2}$$

where l_{ij} is again the shortest geometric distance between i and j. For an airline network, a better approximation is

$$Z_2 = \sum_{i<j} h_{ij}, \tag{1.3}$$

where h_{ij} is the minimum number of legs in the journey. The total cost of running the network is then proportional to the sum $T + \gamma Z_1$ or $T + \gamma Z_2$, respectively, with $\gamma \geq 0$ a constant that measures the relative importance of the two terms. The optimal network in our model is the one minimizing the thus defined total cost [6, 17].[1]

The number of edges in the network depends on the parameter γ. If $\gamma \to 0$, the cost of travel $\propto Z_{1/2}$ vanishes and the optimal network is the one that simply minimizes the total length of all edges. That is, it is the minimum spanning tree (MST), with exactly $n-1$ edges between the n vertices. Conversely, if $\gamma \to \infty$ then $Z_{1/2}$ dominates the optimization, regardless of the cost T of maintaining the network, so that the optimum is a fully connected network or clique with all $\frac{1}{2}n(n-1)$ possible edges present. For intermediate values of γ, finding the optimal network is a non-trivial combinatorial optimization problem, for which we can derive good, though usually not perfect, solutions using the method of simulated annealing [11].

[1] The critical reader might have noticed that Z_1 has the dimension of a length whereas Z_2 is dimensionless. In this paper, we will get rid of Z_1's dimension by setting the average Euclidean "crow flies" distance between a vertex and its nearest neighbor equal to one. This will be accomplished by placing n vertices in a square of length $2\sqrt{n}$ and imposing periodic boundary conditions.

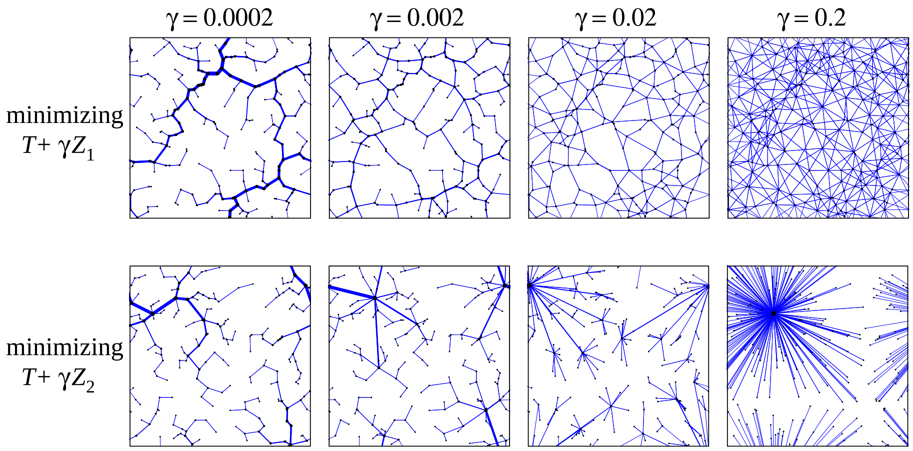

Figure 1.1: Networks minimizing $T + \gamma Z_1$ (top) and $T + \gamma Z_2$ (bottom) for different values of γ and $n = 200$ vertices each. The networks in the top row are obtained by minimizing *geometric* distances between vertices; the bottom row shows the results if the relevant distance for the user is the *graph* distance. The thickness of the edges represents the betweenness defined in Sec. 1.3. Note that we have imposed periodic boundary conditions, i.e. a line leaving the square at the top enters the square again at the bottom, and similarly a line at the left end reappears on the right.

We show networks obtained in this manner in Fig. 1.1. For $\gamma = 0.0002$ the optimum networks are almost identical to MSTs independent of the users' preference for either short mileage or a small number of stopovers. As γ increases, however, the two models show very distinct behaviors. In the first case the number of edges grows, whereas in the second case the networks remain trees for all but very large γ. If users wish to minimize graph distances, a small number of highly connected vertices appear with increasing γ. Like hubs in an airline network, these vertices collect most of the traffic from other vertices in the vicinity. On the other hand, if users care about geometric distances, there are no such hubs. These differences influence the traffic patterns as we will now show.

1.3 Edge betweenness as a measure for traffic flow

The definitions of Z_1 and Z_2, Eq. 1.2 and 1.3, imply three assumptions. First, there is an equal demand for traveling between all origin-destination pairs. Second, all edges have infinite capacities, so that there are no delays due to congestion. Third, all the traffic is along the shortest paths through the network, either measured by geometric or graph distance; in other words, users do not take in-

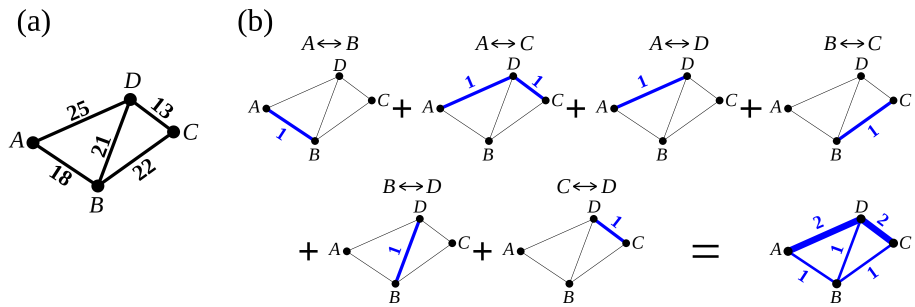

Figure 1.2: Calculation of the edge betweenness. (a) A simple illustrative network. Numbers refer to Euclidean edge lengths. (b) Edge betweennesses for the same network. For every pair of vertices, we send one unit of flow along the shortest geometric path between them. The amount of flow is indicated by bold numbers on the edges. Since there is no edge between A and C, the shortest path between these two vertices is $A \leftrightarrow D \leftrightarrow C$ which is slightly shorter than $A \leftrightarrow B \leftrightarrow C$. Therefore, the edges $A \leftrightarrow D$ and $C \leftrightarrow D$ have a betweenness of 2, all other edges a betweenness of 1. We could have also used graph distance instead of geometric distance, which amounts to setting all distances in (a) equal to one, but this generally gives different results. In terms of graph distance, there are, for example, two shortest paths between vertices A and C, namely via B and via D. Both paths would contribute one half unit of flow to the edge betweenness.

tentional detours. The situation in real networks is, of course, more complicated, but these assumptions are a plausible starting point.

An appropriate way to measure traffic flow under these assumptions is a generalization of the "edge betweenness" which was first introduced in [14]: We send one unit of flow between every possible origin-destination pair along the shortest path and count the number of units that have passed through one particular edge. Equivalently, the edge betweenness is the number of shortest paths in the network running along that edge. In [14] distances are measured as graph distances, but if the user measures path lengths as geometric distances, we can generalize the idea in an obvious manner. A sample calculation is shown in Fig. 1.2.

For the models of Eq. 1.2 and 1.3 we have constructed optimal networks for $n = 200$ randomly placed vertices and measured the betweenness of all edges for several values of γ. In Fig. 1.3, we plot the cumulative distributions, i. e. the fraction of the edges in the network whose betweenness is larger than a certain value b. Panel (a) shows the result for the first model where the user cost Z_1 depends on geometric distances; panel (b) shows the distribution for our second model with the user cost Z_2 depending on graph distances. For $\gamma \rightarrow 0$, where the optimal networks are MSTs, both models possess a long-stretched tail indicating that in this case some edges, like main arteries, have to support a large portion of the flow in the network. If such an edge fails, for example because of

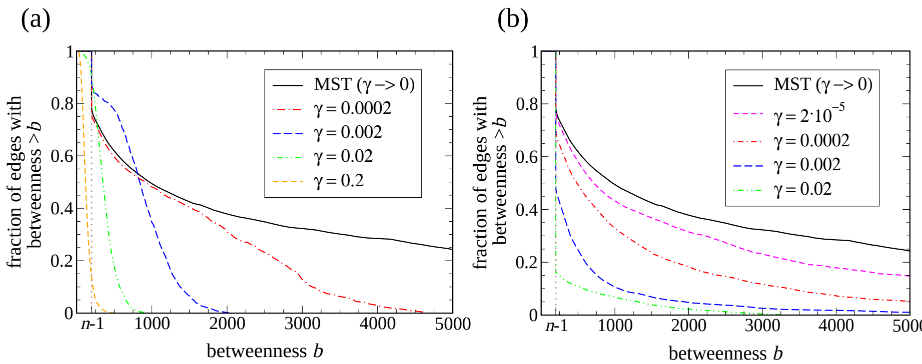

Figure 1.3: Cumulative edge betweenness distributions for networks with $n = 200$ randomly placed vertices. (a) Distributions for networks minimizing the total cost $T + \gamma Z_1$ where the user costs depend on geometric distances. (b) Distributions for networks minimizing the total cost $T + \gamma Z_2$ where the user costs depend on graph distances.

construction or congestion, many routes in the network will be affected.

The distributions for both models become narrower as γ increases. The effect, however, is much stronger in the first than in the second model. For $\gamma = 0.02$, for example, no edge in the first model has a betweenness larger than 1200, whereas in the second model the maximum is around 3300. This difference is closely related to the different network structures. As pointed out in Sec. 1.2, the second model, unlike the first, possesses a small number of highly connected vertices. These hubs collect most of the traffic and, since the networks are trees, the traffic must inevitably pass through the few edges between the hubs which explains their high betweenness. Networks generated by the first model, on the other hand, have no hubs but more edges so that the maximum betweenness is smaller.

Towards the left-hand side most curves in Fig. 1.3 have jumps at $b = n - 1$. These jumps are present if a large fraction of the vertices have a degree of one because these vertices can only be reached along one edge, so that traffic from all other $n-1$ vertices must go through that edge. Since the second model leads to increasingly many such "dead ends" as γ grows, the jumps in Fig. 1.3(b) become bigger. However, a smaller betweenness than $n - 1$ is possible as the curve for $\gamma = 0.2$ in Fig. 1.3(a) proves. For $\gamma \to \infty$ the optimal networks contain all $\frac{1}{2}n(n-1)$ possible edges, hence every edge has in this limit a betweenness equal to one.

1.4 Conclusion

In this paper we have studied the traffic distribution in a spatial network model. The model is based on the optimization of maintenance costs measured by the length of all edges and the ease of travel measured by the sum of all distances between vertex pairs. A single parameter γ determines the relative weight of both considerations. If users prefer short geometric distances, more edges are added to the network. On the other hand, if the user prefers short graph distances, a hub-and-spoke network emerges with only few additional edges.

The traffic along one edge can be measured as its "betweenness" which is the number of shortest paths in the network using this edge. The cheapest network to maintain, the MST, has a small number of edges with very high betweenness. If more weight is given to user-friendliness, the highest betweenness in the network decreases. The effect, however, is stronger if we minimize geometric rather than graph distance. In the first case, the additional edges can reduce the traffic by a large extent, whereas in the second case, the connections between the hubs still carry a substantial amount of traffic.

In our model, we assumed that all edges can in principle handle infinitely much traffic. For future work, one could consider edges with finite capacities so that some edges along the shortest path might become congested and hence unavailable. This problem, however, possesses some non-trivial features [7] requiring a more careful analysis of the users' strategies.

1.5 Acknowledgments

Helpful discussions with M. E. J. Newman are acknowledged.

Bibliography

[1] Albert, R., and A.-L. Barabási, "Statistical mechanics of complex networks", *Reviews of Modern Physics* **74** (2002), 47.

[2] Alvarez-Hamelin, J. I., and N. Schabanel, "An internet graph model based on trade-off optimization", *European Physical Journal B* **38** (2004), 231–237.

[3] Barrat, A., M. Barthélemy, and A. Vespignani, "The effects of spatial constraints on the evolution of weighted complex networks", *Journal of Statistical Mechanics* (2005), P05003.

[4] Barthélemy, M., "Crossover from scale-free to spatial networks", *Europhysics Letters* **63** (2003), 915–921.

[5] Barthélemy, M., and A. Flammini, "Optimal traffic networks", Preprint physics/0601203 (2006).

[6] Billheimer, J. W., and P. Gray, "Network design with fixed and variable cost elements", *Transportation Science* **7** (1973), 49–74.

[7] Braess, D., "Über ein Paradoxon der Verkehrsplanung", *Unternehmensforschung* **12** (1968), 258–268.

[8] Dorogovtsev, S. N., and J. F. F. Mendes, "Evolution of networks", *Advances in Physics* **51** (2002), 1079.

[9] Fabrikant, A., E. Koutsoupias, and C. H. Papadimitriou, "Heuristically optimized trade-offs: A new paradigm for power laws in the internet", *ICALP*, vol. 2380 of *Lecture Notes in Computer Science*, Springer (2002), 110–112.

[10] Flaxman, A. D., A. M. Frieze, and J. Vera, "A geometric preferential attachment model of networks", Preprint http://www.math.cmu.edu/ af1p/GeoWeb.pdf (2006).

[11] Gastner, M. T., *Spatial Distributions: Density-equalizing map projections, facility location, and two-dimensional networks*, PhD thesis, University of Michigan, Ann Arbor (2005).

[12] Gastner, M. T., and M. E. J. Newman, "Shape and efficiency in spatial distribution networks", *Journal of Statistical Mechanics* (2006), P01015.

[13] Gastner, M. T., and M. E. J. Newman, "The spatial structure of networks", *European Physical Journal B* **49** (2006), 247–252.

[14] Girvan, M., and M. E. J. Newman, "Community structure in social and biological networks", *Proceedings of the National Academy of Sciences in the United States of America* **99** (2002), 7821–7826.

[15] Guimerà, A., and L. A. N. Amaral, "Modeling the world-wide airport network", *European Physical Journal B* **38** (2004), 381–385.

[16] Kaiser, M., and C. C. Hilgetag, "Spatial growth of real-world networks", *Physical Review E* **69** (2004), 036103.

[17] Los, M., and C. Lardinois, "Combinatorial programming, statistical optimization and the optimal transportation network problem", *Transportation Research-B* **16B** (1982), 89–124.

[18] Manna, S. S., and P. Sen, "Modulated scale-free network in Euclidean space", *Physical Review E* **66** (2002), 066114.

[19] Newman, M. E. J., "The structure and function of complex networks", *SIAM Review* **45** (2003), 167.

[20] Petermann, T., and P. De Los Rios, "Physical realizability of small-world networks", *Physical Review E* **73** (2006), 026114.

[21] ROZENFELD, A. F., R. COHEN, D. ben AVRAHAM, and S. HAVLIN, "Scale-free networks on lattices", *Physical Review Letters* **89** (2002), 218701.

[22] SEN, P., K. BANERJEE, and T. BISWAS, "Phase transitions in a network with a range-dependent connection probability", *Physical Review E* **66** (2002), 037102.

[23] SEN, P., and S. S. MANNA, "Clustering properties of a generalized critical Euclidean network", *Physical Review E* **68** (2003), 026104.

[24] WARREN, C. P., L. SANDER, and I. M. SOKOLOV, "Geography in a scale-free network model", *Physical Review E* **66** (2002), 056105.

[25] XULVI-BRUNET, R., and I. M. SOKOLOV, "Evolving networks with disadvantaged long-range connections", *Physical Review E* **66** (2002), 026118.

[26] YOOK, S.-H., H. JEONG, and A.-L. BARABÁSI, "Modeling the internet's large-scale topology", *Proceedings of the National Academy of Sciences of the United States of America* **99** (2002), 13382–13386.