A Dynamic Theory of Strategic Decision Making applied to the Prisoner’s Dilemma

Gerald H. Thomas and Keelan Kane
Milwaukee School of Engineering
Adapt of Illinois

1. Introduction

The classic prisoner’s dilemma (PD) game has been extensively investigated by game theorists since the late 1950s, and has been scrutinized in both theoretical and empirical contexts. It has been concluded by many that the Nash equilibrium does not apply to this game. Here we reexamine the issue from the perspective of a dynamic theory of strategic decision making constructed along the lines of physics (Thomas 2006) and show from this larger perspective that the Nash equilibrium must be extended to include dynamic possibilities. We show that this formulation leads one to recognize that interactions simultaneously involve both self-interest and the interests of others, even if one starts by adopting the assumption that agents are driven only by self-interest or only by other-interest. This result has consequences far beyond the simple example of the prisoner’s dilemma.

The dynamic view is extraordinarily rich and provides a strategy for examining general decision making processes. We note that the approach is based on observed behaviors (strategies) and observed outcomes (utilities). As such it is subject to direct observation and refinement using the scientific method. We use the nomenclature egoists (self-interest) and altruists (other-interest) as described for example by Eshel et al. (1998) and combine it with the physics framework in order to examine the PD game. This paper begins with a formulation of the Prisoner’s Dilemma, a description of the physical framework, followed by a description of the PD game in the same terms.

2. Prisoner's Dilemma Formulation

The prisoner's dilemma is a game between two players/prisoners specified by a payoff matrix for each player. The payoff matrix for player 1 is:

G_{12}	N_2	C_2
N_1	-0.1	-1
C_1	0	-0.9

There is an identical payoff matrix for player 2. Using the standard game theory analysis we see that if player 1 chooses N_1 to not confess, the worst that can happen is that player 2 confesses C_2 and the payoff is -1.0. If however player 1 chooses C_1 to confess, the worst that can happen is that player 2 again confesses C_2 with a payoff of -0.9. Since this last choice has a better payoff, the standard game theory analysis concludes that player 1 confesses. Player 2 sees exactly the same game matrix, and so also confesses. The identified solution is called the Nash equilibrium. It is the game theory optimal solution despite the dilemma that if each player were not to confess they would both be better off. Indeed, the latter also squares better with what some of would expect.

To apply the new dynamic theory of decisions, we first frame the game each player sees as an equivalent *symmetric* zero-value game; because there are two distinct payoffs the overall game is a non-zero sum game. Each player's view is of a zero-sum game whose expected payoff is -0.1 units. The equivalent symmetric game is a three player game: two symmetric players whose strategic choices are the union of the original two players' strategies, and a third "hedge" player with a single strategy whose payoffs are such to insure the new game has a zero payoff if the optimal strategies of the original game are chosen. An identical equivalent game exists for player 2. The payoff matrix when multiplying the equilibrium strategy $\xi = \{0 \quad 1 \quad 0 \quad 1 \quad m\}$ yields zero. The relative scale for the hedge strategy m does not affect the strategic outcome. The symmetric game payoff is the antisymmetric matrix:

F^1_{ab}	N_2	C_2	N_1	C_1	H
N_2	0	0	0.1	0	0
C_2	0	0	1.0	0.9	$-\frac{1}{m}0.9$
N_1	-0.1	-1.0	0	0	$\frac{1}{m}1.0$
C_1	0	-0.9	0	0	$\frac{1}{m}0.9$
H	0	$\frac{1}{m}0.9$	$-\frac{1}{m}1.0$	$-\frac{1}{m}0.9$	0

Symmetric games are equivalent to linear programming problems, and admit to simple numerical analysis. Symmetric games can also be solved as coupled differential equations (for example, see Luce and Raiffa):

$$\frac{d\mathbf{V}}{d\tau} = \mathbf{F} \cdot \mathbf{V}$$

This differential equation will be the starting point for a dynamic theory. This equation describes *stationary* behavior if the flow vector $\mathbf{V}$ does not change in time, which occurs when the right hand side vanishes. The dynamic notion of stationary flow replaces the game theory notion of equilibrium. The differential equation also describes *dynamic* behavior that is not stationary.

3. Egoists and altruists in a dynamic theory

3.1 Thomas' Dynamic Theory of Strategic Decisions

Thomas (2006) takes the differential equation form of the game theory seriously as representing the game, identifying the hedge strategy with time. A general form of the differential equation is postulated, taking into account the geometry of the strategy space through the *active* geometry metric elements g_{ab} and *inactive* geometry metric elements $\gamma^{\alpha\beta}$. The metric elements provide the method for specifying distance between two plays of the same game. If the metric elements are independent of a strategy, it is considered inactive; otherwise it is active. Strategies are defined as in standard game theory with the game in its extensive form, and with an antisymmetric decision matrix F^{α}_{ab} for each player α. In addition to strategies for each player, there is the outcome or utility for that player: the hypothesis is that the outcome is an inactive dimension of the geometry. Successive plays of the same game represent a "flow" in this geometry. For active strategies, the flow is represented as V^a; for inactive strategies the flow is considered a "charge" and represented by V_α. The behavior of a new play of a game is determined by those games already played by a set of deterministic causal equations in which pressure p and matter density μ have been ascribed to the game.

The full set of equations is the economic version of Einstein's equations applied to this geometry. The resultant flow equations, the economic version of Euler's equations, result from the conservation of energy and momentum:

$$g_{ab}\frac{DV^b}{\partial\tau} = V_\alpha F^{\alpha}_{ab}V^b - \tfrac{1}{2}V_\alpha V_\beta \partial_a \gamma^{\alpha\beta} + h^b_a \frac{\partial_b p}{\mu + p}.$$

These equations replace the differential equation from the last section with the weighted sum $V_\alpha F^{\alpha}_{ab}$, from each player's charge times its payoff, identified as the zero-sum symmetric zero-value game $\mathbf{F}$. The sum $V_\alpha F^{\alpha}_{ab}$ represents the composite payoff that determines the behavior of the game, and will in general be quite different from the separate payoffs for non-zero sum games unless each player sees the same zero-sum game.

There are three important consequences of the above equations: any definition of equilibrium is dynamic and based on the flows being stationary; the "game" aspect of the equation is represented as the specific composite sum $V_\alpha F^{\alpha}_{ab}$; and there are non-game aspects that influence the dynamics. Moreover, in the composite sum, the coefficients are not arbitrary, but are themselves governed by equations that also derive from the economic Einstein equations, and are the analogs of Maxwell's equations:

$$\tfrac{1}{2}\frac{1}{\sqrt{|g\gamma|}}\partial_b\left(\sqrt{|g\gamma|}g^{ac}g^{bd}\gamma_{\alpha\beta}F^{\beta}_{cd}\right)=\kappa\left(\mu+p\right)V_\alpha V^a .$$

These equations show that the current, the product of the "charge" V_α , flow V^a , matter density $\mu+p$ and coupling constant κ , determine the player decision matrix F^{α}_{ab} .

The form of these Maxwell equations have profound implications that depend on the number of active dimensions of the geometry, and split into two distinct sets of equations. The first set, the time component of the equations, has the form of "Coulomb's" Law $\nabla\cdot\mathbf{E}=\kappa j^0_\alpha$. The important consequence of this is that this equation implies that like charges repel and opposite charges attract. This equation does not involve second order time derivatives, and is a constraint equation that must be satisfied as an initial condition. If like charges are nearby at the start, the form of the equations implies that they move away from each other as time increases. For this reason we argue that systems comprised of only egoist players (or only altruist players) are expected to fly apart. Matter as a rule consists of equal mixtures of charges separated by short distances. We identify the positive charges with altruistic behavior, negative charges with egoist behavior and equal mixtures with normal behavior. We assert that normal behavior is likely composed of equal amounts of altruist and egoist behavior.

The second set, the space components of the equations, has the form analogous to the Biot-Savart Law $\nabla\times\mathbf{B}_\alpha+\partial\mathbf{E}/\partial t=\kappa\mathbf{j}_\alpha$ in which currents generate time changing magnetic and electric fields. For a general dimension of the geometry, such equations can be converted to wave equations involving the second order time derivatives of a "vector" potential. Such waves radiate with a fixed and finite velocity whenever the active space-time dimension D_A is greater than two. In such cases the number of polarization states for the radiation is D_A-2 . In physics, this active dimension is four, resulting in two possible polarization states; such radiation is called "light" or photons. For the prisoner's dilemma example, the total dimension of the space is seven with two inactive dimensions (the utilities or outcomes), so the active dimension is five or less. For systems that are not static (time is active) and in which each player has at least one active strategy, then the active dimension is three or greater. There will then be radiation and this "radiation" determines which past events can influence any given event.

The radiation is an intrinsic property of the strategic decision fields of each player, is a consequence of the theory being a gauge theory, and its confirmation would provide strong support for this physical approach. In particular if a player consists of opposite charges separated by some distance circling each other, then there would be acceleration and hence radiation leading to the charges collapsing onto each other. In physics, this is prevented by adopting the quantum wave view of matter. We thus have some striking consequences that extend far beyond the prisoner's dilemma game.

3.2 Composite Fields, Charges and Null Behaviors

If a game is played the same way over and over, the game creates a "current" for each player that through the homologous Maxwell equation above creates the decision field F^{α}_{ab}. A test play of the game will then see a force through the homologous Euler flow equations. Since the equations are highly non-linear, we believe it is helpful to think of these equations sequentially to describe how they work.

In this conceptualization, the force based on the decision for a given player is determined by the size of $V_{\alpha}F^{\alpha}_{ab}V^{b}$. The size can be judged against the "direction" of the decision field defined as the null vector ξ^{b} defined as satisfying $F^{\alpha}_{ab}\xi^{b}=0$. If the flow is along the null direction there is no force; the null direction provides the direction of the static equilibrium. It can be shown that non-equilibrium motion occurs in a helix around this direction, with a direction of rotation and a frequency given by the strength of the charge density. There is a decision matrix for each player, and the corresponding null vectors are in general not equal.

We use these two concepts below in our analysis of the prisoner's dilemma: for each player we identify the null direction ξ^{b}_{α} and the charge V_{α}. We emphasize that the null direction is determined by the sources that generate it, and so is not independent of the charge. In a complete analysis, we would solve the coupled equations simultaneously. For this paper, we simply illustrate the possibilities by considering appropriate special cases.

4. Prisoner's Dilemma Analysis

There are two charges—altruist (positive) and egoist (negative). In addition, for each prisoner there are two null behaviors depending on whether the decision matrix derived from the Maxwell equations results from an altruist or egoist charge. There are sixteen cases in all. We create the composite $V_{\alpha}F^{\alpha}_{ab}$ by taking the sum of the player decision matrices weighted by their player's charge assuming that a player is either altruistic or egoistic. The sixteen possible cases can be reduced to the following four cases:

- There is a composite behavior constructed from two egoists (egoistic null behavior);
- There is a composite market constructed from two altruists (altruistic null behavior);
- There is composite market constructed from an altruist and an egoist where each acts appropriately or each acts oppositely to their null behavior;
- And there is a composite market constructed from an altruist and an egoist where one acts appropriately and the other oppositely.

4.1 Egoistic null behavior from two egoists

We believe the composite market field for the prisoner's dilemma, as usually stated reflects an egoist null behavior. In this and the other examples, we compute $V_{\alpha}F^{\alpha}_{ab}$

assuming that the charge for an egoist is –1, for an altruist is +1, and that the egoist null behavior is that given in the earlier section. With this in mind the composite field is made up of two identical payoff matrices. If both players are motivated by self-interest, they both confess. If the players' utilities were comparable, such as expressible in for example in dollars, the sum of the payoffs is -1.8 units, significantly less than the penalty of -0.2 if they cooperated. The composite matrix is:

$$V_\alpha F^\alpha_{ab} = -\begin{pmatrix} 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 1 & 0.9 & -{}^{0.9}\!/_m \\ -0.1 & -1 & 0 & 0 & {}^{1}\!/_m \\ 0 & -0.9 & 0 & 0 & {}^{0.9}\!/_m \\ 0 & {}^{0.9}\!/_m & -{}^{1}\!/_m & -{}^{0.9}\!/_m & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -0.1 & -1 & {}^{1}\!/_m \\ 0 & 0 & 0 & -0.9 & {}^{0.9}\!/_m \\ 0.1 & 0 & 0 & 0 & 0 \\ 1 & 0.9 & 0 & 0 & -{}^{0.9}\!/_m \\ -{}^{1}\!/_m & -{}^{0.9}\!/_m & 0 & {}^{0.9}\!/_m & 0 \end{pmatrix}$$

The dilemma is that this does not correspond to our experience. We know that prisoners, such as Mafiosi, may hold a great deal of loyalty to others, and will not in general confess. The argument is even stronger if we consider political prisoners. Nevertheless, for this case, the null vector of the composite is that each player confesses since this is the null vector of each term.

4.2 Altruistic null behavior from two altruists

An altruistic player differs from an egoistic player in that such a player sees a different utility and market. Let's start with a purely egoistic world and appeal to the homologous Maxwell equations to obtain guidance for the form in a purely altruistic world. If the sign of the charge changes, and the market fields (i.e. the notion of utility) keep the same signs, then the flows will in general be reversed. To keep the flow positive, it suggests we change the sign of the utility. This has the intuitive appeal that the altruist makes the same type of min-max argument that the egoist does, but the altruist player takes their utilities to be opposite in sign. The composite market field is made up of what each player sees:

$$V_\alpha F^\alpha{}_{ab} = \begin{pmatrix} 0 & 0 & -0.1 & 0 & {}^{0.1}\!/_m \\ 0 & 0 & -1 & -0.9 & {}^{1}\!/_m \\ 0.1 & 1 & 0 & 0 & -{}^{0.1}\!/_m \\ 0 & 0.9 & 0 & 0 & 0 \\ -{}^{0.1}\!/_m & -{}^{1}\!/_m & {}^{0.1}\!/_m & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0.1 & 1 & -{}^{0.1}\!/_m \\ 0 & 0 & 0 & 0.9 & 0 \\ -0.1 & 0 & 0 & 0 & {}^{0.1}\!/_m \\ -1 & -0.9 & 0 & 0 & {}^{1}\!/_m \\ {}^{0.1}\!/_m & 0 & -{}^{0.1}\!/_m & -{}^{1}\!/_m & 0 \end{pmatrix}.$$

The first term is player 1. Such a player would look at the possibilities as follows: if she does not confess, the worst that can happen is if player 2 does not confess, and she gains 0.1 units; if she confesses, the worst that can happen is that player 2 does not confess, and she gains 0 unit. Of these two cases the best for her is not to confess. The optimal strategy as she sees it is $\{1 \quad 0 \quad 1 \quad 0 \quad m\}$. Player 2 sees the same possibilities. Thus this is the opposite of two egoists.

If both players are altruistic, they both remain silent. The composite market field yields the same result. The composite game is altruistic, both players choose to remain silent, and the total payoff if their utilities were compatible is 0.2. We obtain the important result that with two altruistic players, game theory applies using the usual rules for computing the null behavior if we allow the utilities to reflect altruistic charge.

4.3 Composite game with egoist and altruist where each acts appropriately (or both oppositely) to their behavior

In the two previous cases, the null behavior of the composite game does not depend on the charges of the players. If an egoist played as if she were an altruist (opposite charge), then only the coefficient of the composite null behavior would change resulting in the same equilibrium value. If we have an egoist null behavior for player 1 and altruist null behavior for player 2, then the composite null behavior depends on the relative sign and value of the charges. To keep things simple we continue to focus on unit charges, and allow the charges to be either +1 or –1. Thus in the two previous cases, we were able to deduce the composite null behavior by taking the null behavior of each player individually.

In the remaining cases, the composite null behavior need not be the null behavior of any one player. The null behavior is the stationary direction ξ^b along which the composite market produces no force, $V_\alpha F^\alpha_{ab}\xi^b$. We find the composite null behavior by finding those vectors in the null space of the composite matrix. The null vectors can be found using standard techniques and verified by inspection. We provide only the answer here, and the reader can easily verify their correctness.

We start by taking the charge for player 1 to be –1 and the charge for player 2 to be +1 corresponding to the market that they see respectively. The composite field is then:

$$V_\alpha F^\alpha{}_{ab} = -\begin{pmatrix} 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 1 & 0.9 & -{}^{0.9}\!/_m \\ -0.1 & -1 & 0 & 0 & {}^{1}\!/_m \\ 0 & -0.9 & 0 & 0 & {}^{0.9}\!/_m \\ 0 & {}^{0.9}\!/_m & -{}^{1}\!/_m & -{}^{0.9}\!/_m & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0.1 & 1 & -{}^{0.1}\!/_m \\ 0 & 0 & 0 & 0.9 & 0 \\ -0.1 & 0 & 0 & 0 & {}^{0.1}\!/_m \\ -1 & -0.9 & 0 & 0 & {}^{1}\!/_m \\ {}^{0.1}\!/_m & 0 & -{}^{0.1}\!/_m & -{}^{1}\!/_m & 0 \end{pmatrix}.$$

We have taken the form for player 2 from the previous section. Neither the pure egoist nor the pure altruist solution is an equilibrium value. The composite decision matrix is:

$$V_\alpha F^\alpha{}_{ab} = \begin{pmatrix} 0 & 0 & 0 & 1 & -{}^{0.1}\!/_m \\ 0 & 0 & -1 & 0 & {}^{0.9}\!/_m \\ 0 & 1 & 0 & 0 & -{}^{0.9}\!/_m \\ -1 & 0 & 0 & 0 & {}^{0.1}\!/_m \\ {}^{0.1}\!/_m & -{}^{0.9}\!/_m & {}^{0.9}\!/_m & -{}^{0.1}\!/_m & 0 \end{pmatrix}$$

The null behavior is $\{0.1 \quad 0.9 \quad 0.9 \quad 0.1 \quad m\}$. The altruist (player 2) predominantly chooses the option to confess (with odds of 9:1) and the egoist (player 1) predominantly chooses the option to not confess (with odds 9:1). Each player is impacted significantly because of the presence of the oppositely behaving player If each player were to act oppositely to their market, then the charges would be opposite to the calculation above. Though it changes the direction of motion around the null direction, it does not change the direction.

4.4 Composite game with egoist and altruist where one acts appropriately and the other oppositely

We enumerate the total cases in the following table, where the rows and columns label player 1 and 2 respectively, E and A represent egoist and altruist, and M_E and M_A represent the egoist or altruist null behavior the player sees:

$1/2$	$E\otimes M_E$	$E\otimes M_A$	$A\otimes M_E$	$A\otimes M_A$
$E\otimes M_E$	X_1	X_4	X_1	X_3
$E\otimes M_A$	X_4	X_2	X_3	X_2
$A\otimes M_E$	X_1	X_3	X_1	X_4
$A\otimes M_A$	X_3	X_2	X_4	X_2

Of these 16 total possibilities there are four distinct cases of which we have considered the egoistic null behavior with two egoists (X_1), the altruistic null behavior with two altruists (X_2), and the composite game with one egoist null behavior and one altruistic null behavior with both acting the same (or opposite) to their null behavior (X_3). There remains one case X_4 that is distinct from the others.

We select as representative an egoist null behavior and an altruistic null behavior where both players are egoists (X_4). The composite market is:

$$V_\alpha F^\alpha{}_{ab} = -\begin{pmatrix} 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 1 & 0.9 & -{}^{0.9}\!/_m \\ -0.1 & -1 & 0 & 0 & {}^{1}\!/_m \\ 0 & -0.9 & 0 & 0 & {}^{0.9}\!/_m \\ 0 & {}^{0.9}\!/_m & -{}^{1}\!/_m & -{}^{0.9}\!/_m & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0.1 & 1 & -{}^{0.1}\!/_m \\ 0 & 0 & 0 & 0.9 & 0 \\ -0.1 & 0 & 0 & 0 & {}^{0.1}\!/_m \\ -1 & -0.9 & 0 & 0 & {}^{1}\!/_m \\ {}^{0.1}\!/_m & 0 & -{}^{0.1}\!/_m & -{}^{1}\!/_m & 0 \end{pmatrix}.$$

The composite game is:

$$V_\alpha F^\alpha{}_{ab} = \begin{pmatrix} 0 & 0 & -0.2 & -1 & {}^{0.1}\!/_m \\ 0 & 0 & -1 & -1.8 & {}^{0.9}\!/_m \\ 0.2 & 1 & 0 & 0 & -{}^{1.1}\!/_m \\ 1 & 1.8 & 0 & 0 & -{}^{1.9}\!/_m \\ -{}^{0.1}\!/_m & -{}^{0.9}\!/_m & {}^{1.1}\!/_m & {}^{1.9}\!/_m & 0 \end{pmatrix}$$

The composite null behavior is $\{-1 \quad 9 \quad 9 \quad -1 \quad 8m\}$. The negative strategy may lie outside the allowed range, and a more detailed dynamic analysis is called for. However we suggest a possible interpretation assuming strategies are bounded inside a box characterized by positive strategies. In this case, the dynamic equations operate even though it is not possible to move along that direction. The box provides additional boundary conditions. The negative strategy for the egoist reflects the possibility that he can not play the "confess" strategy indefinitely. This is because the optimal strategy lies outside the box. Similarly, the altruist can't play the "silent" strategy indefinitely. Both players are influenced by the other's behaviors.

5. Summary

We assert that the Nash equilibrium must be extended to include dynamic possibilities. A dynamic theory extends the static standard game theory, and structurally changes the notion and importance of game theory equilibrium. Dynamic theories are characterized by complex behavior and distinguished by notions such as fixed points, *stationary* or constant flow that can be attractive or repulsive, and periodic or semi-periodic behavior. A dynamic theory can display even more complex behavior such as turbulence, chaos or bios [Sabelli 2005, and this conference], each with their own characterizations. Indeed we have observed biotic behavior in the proposed dynamic theory.

Our analysis is based on a causal evolution of behavior from some fixed point in time. Because of the nature of the dynamics we are led to the conclusion that games are not player only by altruists, or indeed only by altruists, but by players reflecting an equal mixture of both attributes (charges). To the extent that such charges occupy nearby spaces, it may be that some "quantum" formulation is necessary for the theory to be consistent.

The focus of this paper has been on a behaviorist view of the prisoner's dilemma, and on how to articulate the behavior inside a dynamic theory. There are many other interesting questions that relate to the prisoner's dilemma, and in the process of studying that problem we have collected a few preliminary though by no means complete thoughts on these questions.

Although the prisoner's dilemma originated from efforts to anticipate whether suspects arrested for particular crimes would confess or remain silent (hence the game's name), the "prisoner's dilemma" framework has been generalized and redesigned to investigate other phenomena that rely on similar reasoning—e.g., whether people will pollute the environment or will dispose of waste properly (see, e.g., Camerer, 2003, Chapter 2).

There are empirical investigations of human participants presented with prisoner dilemma game situations, which have yielded results suggesting that other contexts in which the game is played affect players' decisions.

Some of the phenomena observed in empirical studies have motivated "psychological" approaches to modeling the game (e.g., Dufwenberg & Kirchsteiger, 1998; Rabin, 1993). The hallmark of these psychological frameworks is that they attempt to model players' "fairness" or "kindness,", as well as each player's *beliefs* about whether his or her own actions will be reciprocated with (un)kind or (un)fair actions. As Rabin (1993) notes in his psychological model, the notion of *altruism* can bear on the notion of fairness: "*the same people who are altruistic to other altruistic people are also motivated to hurt those who hurt them"* (p. 1281, emphasis in original).

References

Camerer, Colin F. (2003) *Behavioral Game Theory: Experiments in Strategic Thinking.* Russell Sage Foundation: New York, NY.

Dufwenberg, Martin & Kirchsteiger, Georg (1998). A theory of sequential reciprocity. Tilburg Center for Economic Research discussion paper 9837.

Eshel, Ilan; Samuelson, Larry; & Shaked, Avner (1998). Altruists, egoists, and hooligans in a local interaction model. *The American Economic Review*, March, 157-179.

Luce, R. D., and Raiffa, H. (1957), *Games and Decisions* (Dover Publications, NY).

Rabin, Matthew (1993). Incorporating fairness into game theory and economics. *American Economic Review, 83*, 1281-1302.

Sabelli, Hector, *Bios, a Study of Creation* (World Scientific, 2005).

Thomas, Gerald H., *Geometry, Language and Strategy* (World Scientific. 2006).