

## Chapter 1

# A new measure of heterogeneity of complex networks based on degree sequence

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Many unique properties of complex networks are due to the heterogeneity. The measure and analysis of heterogeneity is important and desirable to research the behaviours and functions of complex networks. In this paper, entropy of degree sequence (EDS) as a new measure of the heterogeneity of complex networks is proposed and normalized entropy of degree sequence (NEDS) is defined. EDS is agreement with the normal meaning of heterogeneity within the context of complex networks compared with conventional measures. The heterogeneity of scale-free networks is studied using EDS. The analytical expression of EDS of scale-free networks is presented by introducing degree-rank function. It is demonstrated that scale-free networks become more heterogeneous as scaling exponent decreases. It is also demonstrated that NEDS of scale-free networks is independent of the size of networks which indicates that NEDS is a suitable and effective measure of heterogeneity.

## 1 Introduction

We are surrounded by networks. Networks with complex topology describe a wide range of systems in nature and society [1, 2].

In late 1950s, Erdős and Rényi made a breakthrough in the classical mathematical graph theory. They described a network with complex topology by a random graph (ER model)[3]. In the past few years, the discovery of small-world[4] and scale-free properties[5] has stimulated a great deal of interest in studying the underlying organizing principles of various complex networks.

There are major topological differences between random graphs and scale-free networks. For random networks, each vertex has approximately the same degree  $k \approx \langle k \rangle$ . In contrast, the scale-free network with power-law degree distribution implies that vertices with only a few edges are numerous, but a few nodes have a very large number of edges. The presence of few highly connected nodes (i.e. 'hubs') is the most prominent feature of scale-free networks and indicates that scale-free networks are heterogeneous. The heterogeneity leads to many unique properties of scale-free networks. For example, Albert et al. [6] demonstrated that scale-free networks possess the robust-yet-fragile property, in the sense that they are robust against random failures of nodes but fragile to intentional attacks. Moreover, it is found that homogeneous networks are more synchronizable than heterogeneous ones, even though the average network distance is larger[7]. Consequently, the measure and analysis of heterogeneity is important and desirable to research the behaviours and functions of complex networks.

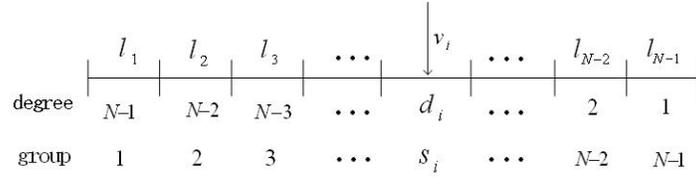
Several measures of heterogeneity have been proposed. Nishikawa et al. [7] quantified the heterogeneity of complex networks using the standard deviation of degree  $\sigma$ . Solé et al.[8] proposed entropy of the remaining degree distribution  $q(k)$  to measure the heterogeneity. Wang et al.[9] measured the heterogeneity of complex networks using entropy of the degree distribution  $P(k)$ . With these measures above, the most heterogeneous network is the network obtained for  $P(1) = P(2) = \dots = P(N - 1)$ , and the most homogeneous network is the network obtained for  $P(k_0) = 1$  and  $P(k) = 1(k \neq k_0)$ , i.e. a regular networks. However, these conventional measures are not in agreement with the normal meaning of heterogeneity within the context of complex networks. For example, we are generally inclined to believe that a random network is quite homogeneous, but it is not the truth with the measure above. In addition, a star network is generally considered to be very heterogeneous because of the presence of the only hub-node, but the star network is quite homogeneous with the conventional measures.

In this paper, we first present a new measure of heterogeneity called entropy of degree sequence (EDS) and compare it with conventional measures. Then we investigate the heterogeneity of scale-free networks using EDS.

## 2 Entropy of degree sequence

A complex network can be represented a graph  $G$  with  $N$  vertices and  $M$  edges. Assume that  $G$  is an undirected and simple connected graph. Let  $d_i$  be the degree of a vertex  $v_i$ . As shown in Figure 1, we sort all vertices in decreasing order of degree and get a degree sequence  $D(G) = (D_1, D_2, \dots, D_N)$ , where  $D_1 \geq$

$D_2 \geq \dots \geq D_N$ . Note that different vertices may have the same degree, we group all vertices into  $N - 1$  groups according to their degree. That is, the degree of the vertices in the  $s^{th}$  group is  $N - s$ . Let  $l_s$  be the number of vertices in the  $s^{th}$  group, namely, the frequencies of vertices with degree  $N - s$ . Let  $s_i$  be the index of the group in which vertex  $v_i$  is located, and  $r_i$  be the global rank of  $v_i$  among all of the vertices in decreasing order of degree. Let  $P(k)$  be the *degree distribution*, i.e. the probability that a randomly chosen vertex has degree  $k$ . Let  $d = f(r)$  be the *degree-rank function*, which gives the relationship function between the degree and the rank of the degree sequence  $D(G)$  and is non-stochastic, in the sense that there need be no assumption of an underlying probability distribution for the sequence.



**Figure 1:** Sorting all vertices in decreasing order of degree and group all vertices into groups according to degree.

To measure the heterogeneity of complex networks, we define entropy of degree sequence (EDS) as

$$EDS = - \sum_{i=1}^N I_i \ln I_i \quad (1)$$

where

$$I_i = D_i / \sum_{i=1}^N D_i \quad (2)$$

Substituting equation (2) into equation (1), we have

$$EDS = - \frac{\sum_{i=1}^N D_i \ln D_i}{\sum_{i=1}^N D_i} + \ln \left( \sum_{i=1}^N D_i \right) \quad (3)$$

Obviously, the maximum value of EDS is  $EDS_{max} = \log(N)$  obtained for  $I_i = 1/N$ , i.e.  $D_1 = D_2 = \dots = D_N$ . Note that  $D_i > 0$  and  $D_i$  is integer, so the minimum value of  $EDS_{min} = (\ln 4(N - 1))/2$  occurs when  $D(G) = (N - 1, 1, \dots, 1)$ . The maximum value of EDS corresponds to the most homogeneous network, i.e. a regular network, and the minimum value of EDS corresponds to the most heterogeneous network, i.e. a star network.

The normalized entropy of degree sequence (NEDS) can be defined as

$$NEDS = \frac{ERD_{\max} - ERD}{ERD_{\max} - ERD_{\min}} \quad (4)$$

For comparison, we present the definition of entropy of remain degree distribution (ERD) in [8]

$$ERD = - \sum_{k=1}^N q(k) \ln q(k) \quad (5)$$

where  $q(k) = (k+1)P(k) / \sum_j jP(j)$ , and entropy of degree distribution (EDD) in [9]

$$EDD = - \sum_{k=1}^N P(k) \ln P(k) \quad (6)$$

For ERD or EDD, the maximum value is  $\log(N-1)$  obtained for  $P(k) = 1/(N-1)$  ( $\forall k = 1, 2, \dots, N-1$ ) which corresponds to the most heterogeneous network and the minimum value is 0 obtained for  $P(k_0) = 1$  and  $P(k) = 0$  ( $k \neq k_0$ ) which corresponds to the most homogeneous network. To be consistent with NEDS, we define the normalized entropy of remain degree distribution (NERD) and the normalized entropy of degree distribution (NEDD) as

$$NERD = \frac{ERD - ERD_{\min}}{ERD_{\max} - ERD_{\min}} \quad (7)$$

and

$$NEDD = \frac{EDD - EDD_{\min}}{EDD_{\max} - EDD_{\min}} \quad (8)$$

Then a network becomes more heterogeneous as NEDS/NERD/NEDD increases.

Using NEDS, NERD, NEDD and standard deviation of degree  $\sigma$  to measure: (a) a regular network with  $N = 1000$ ; (b) a random network (ER model) with  $N = 1000$  and connection probability  $p = 0.3$ ; (c) a star network with  $N = 1000$ ; (d) a scale-free networks (Barabasi-Albert model) with  $N = 1000$  and  $m_0 = m = 3$ , the result is shown in Table. 1.

With NERD or NEDD, the order of heterogeneity is that random network  $\succ$  scale-free network  $\succ$  star network  $\succ$  regular network. With standard deviation of degree  $\sigma$ , the order of heterogeneity is that star network  $\succ$  random network  $\succ$  scale-free network  $\succ$  regular network. With NEDS, the order of heterogeneity is that star network  $\succ$  scale-free network  $\succ$  random network  $\succ$  regular network. We are generally inclined to believe that a scale-free network is more heterogeneous than a random network and a star network is very heterogeneous because of the presence of the only hub-node. So our measure is agreement with the normal meaning of heterogeneity within the context of complex networks compared with conventional measures.

**Table 1:** Result of comparisons between the conventional measures and ours.

	NEDS	NERD	NEDD	$\sigma$
Regular network	0	0	0	0
Random network	0.0004	0.6451	0.6480	26.2462
Scale-free network	0.1178	0.4304	0.2960	6.8947
Star network	1	0.0502	0.0011	31.5595

### 3 Heterogeneity of scale-free networks

To obtain EDS of scale-free networks, we first present a theorem on the relationship between the degree distribution  $P(k)$  and the degree-rank function  $f(r)$ .

**Theorem 1** If the degree distribution of a network is  $P(k)$ , the degree-rank function of the network is  $d = f(r) = N - T^*$ , where  $T^*$  satisfies  $\int_1^{T^*} P(k = N - s) \cdot ds = r/N$ .

PROOF

Since  $v_i$  is located in the  $s_i^{th}$  group, then we obtain

$$\sum_{s=1}^{s_i} l_s \geq r_i \quad \text{and} \quad \sum_{s=1}^{s_i-1} l_s \leq r_i \tag{9}$$

Namely,  $s_i$  is the minimum  $T$  that satisfies  $\sum_{s=1}^T l_s \geq r_i$ , i.e.  $s_i = T_{min}(\sum_{s=1}^T l_s \geq r_i)$ . Note that  $l_s = N \cdot P(k = N - s)$ , we obtain

$$s_i = T_{min}(\sum_{s=1}^T P(k = N - s) \geq r_i/N) \tag{10}$$

Then

$$d_i = N - s_i = N - T_{min}(\sum_{s=1}^T P(k = N - s) \geq r_i/N) \tag{11}$$

Assuming that  $P(k)$  is integrabel, with continuous approximation for the degree distribution, equation. (11) can be written as

$$d_i = N - T_{min}(\int_1^T P(k = N - s)ds \geq r_i/N) \tag{12}$$

Note that  $P(k = N - s) \geq 0$ , hence  $\int_1^T P(k = N - s)ds$  is an increasing function with respect to  $T$ , leading to

$$\int_1^{T_{min}} P(k = N - s)ds = r_i/N \tag{13}$$

Using equation (13), equation (12) can be expressed as

$$d_i = f(r_i) = N - T^* \quad (14)$$

where  $T^*$  satisfies

$$\int_1^{T^*} P(k = N - s) ds = r_i/N \quad (15)$$

The theorem is proved. ■

For scale-free networks with power law degree distributions  $P(k) = Ck^{-\lambda}$ , where  $\lambda$  is the scaling exponent, substituting  $P(k) = Ck^{-\lambda}$  into equation (15), we have

$$\int_1^{T^*} C(N - s)^{-\lambda} ds = r_i/N \quad (16)$$

Solving equation (16) for  $T^*$ , we have

$$T^* = N - \left[ \frac{\lambda - 1}{NC} \cdot r_i + (N - 1)^{-\lambda+1} \right]^{\frac{1}{-\lambda+1}} \quad (17)$$

Substituting equation (17) into equation (14), we obtain the degree-rank functions of scale-free networks as follows:

$$d = f(r) = \left[ \frac{\lambda - 1}{NC} \cdot r + (N - 1)^{-\lambda+1} \right]^{\frac{1}{-\lambda+1}} \quad (18)$$

Note that the scaling exponent of most scale-free networks in real world ranges between 2 and 3. We have  $(N - 1)^{-\lambda+1} \rightarrow 0$  as  $N \rightarrow \infty$  when  $\lambda > 2$ . Then, equation (18) simplifies to

$$d = f(r) \approx C_1 \cdot r^{-\alpha} \quad (19)$$

where  $C_1 = \left(\frac{\lambda-1}{N \cdot C}\right)^{-\alpha}$  and  $\alpha = 1/(\lambda - 1)$ . We call  $\alpha$  the degree-rank exponent of scale-free networks.

Substituting equation (19) into equation (3), we obtain EDS of scale-free networks as follows:

$$EDS = -\frac{\sum_{r=1}^N C_1 r^{-\alpha} \ln C_1 r^{-\alpha}}{\sum_{r=1}^N C_1 r^{-\alpha}} + \ln \left( \sum_{r=1}^N C_1 r^{-\alpha} \right) \quad (20)$$

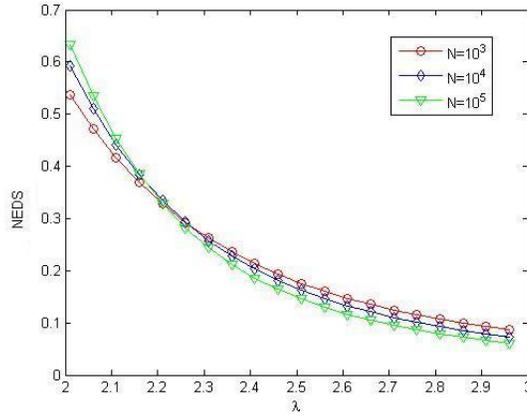
With continuous approximation for the degree distribution, we obtain of scale-free networks as a function of degree-rank exponent  $\alpha$

$$EDS = \frac{\alpha \cdot N^{1-\alpha} \cdot \ln N^{1-\alpha}}{(1-\alpha)(N^{1-\alpha} - 1)} + \ln \frac{N^{1-\alpha} - 1}{1-\alpha} - \frac{\alpha}{1-\alpha} \quad (21)$$

Substituting  $\alpha = 1/(\lambda - 1)$  into equation (21), we obtain EDS of scale-free networks as a function of scaling exponent  $\lambda$

$$EDS = \frac{N^{\frac{\lambda-2}{\lambda-1}} \ln N^{\frac{\lambda-2}{\lambda-1}}}{(\lambda-2)(N^{\frac{\lambda-2}{\lambda-1}} - 1)} + \ln \frac{(\lambda-1)(N^{\frac{\lambda-2}{\lambda-1}} - 1)}{\lambda-2} - \frac{1}{\lambda-2} \quad (22)$$

Substituting Eq. (22) into Eq. (4), we can obtain NEDS of scale-free networks. In Figure 2, we show the plots of NEDS versus for different  $\lambda \in (2, 3)$ . We can find that a scale-free network becomes more heterogeneous as  $\lambda$  decreases. Moreover, we can find that the plots of NEDS for different  $N$  overlap with each other which indicates that NEDS is independent of  $N$  and then NEDS is a suitable measure of heterogeneity for different size of scale-free networks.



**Figure 2:** Plots of NEDS versus  $\lambda \in (2, 3)$  for different  $N$ . It is shown that NEDS decreases as  $\lambda$  increases and NEDS is independent of  $N$ .

## 4 Conclusion

Many unique properties of complex networks are due to the heterogeneity. The measure and analysis of heterogeneity is important and desirable to research the behaviours and functions of complex networks.

In this paper, we have proposed entropy of degree sequence(EDS) and normalized entropy of degree sequence (NEDS) to measure the heterogeneity of complex networks. The maximum value of EDS corresponds to the most heterogeneous network, i.e. star network, and the minimum value of EDS corresponds to the most homogeneous network, i.e. regular network. We measure different networks using conventional measures and ours. The results of comparison have shown that EDS is agreement with the normal meaning of heterogeneity within the context of complex networks.

We have studied the heterogeneity of scale-free networks using EDS and derived the analytical expression of EDS of scale-free networks. We have demonstrated that scale-free networks become more heterogeneous as scaling exponent decreases. We also demonstrated that NEDS of scale-free networks is independent of the size of networks which indicates that NEDS is a suitable and effective measure of heterogeneity.

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