Chapter 1

On Elementary and Algebraic Cellular Automata

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In this paper we study elementary cellular automata from an algebraic viewpoint. The goal is to relate the emergent complex behavior observed in such systems with the properties of corresponding algebraic structures. We introduce algebraic cellular automata as a natural generalization of elementary ones and discuss their applications as generic models of complex systems.

1.1 Introduction

There is a great diversity of complex systems in physics, biology, engineering and other fields that share the ability to exhibit complicated, difficult to predict spatio-temporal behavior. Models of such systems, which reflect their most crucial properties, are based on traditional mathematical approaches, as well as other techniques such as *networks* and *automata*.

In physics and engineering, models can be derived from conservation laws and formulated in terms of differential or difference equations. Due to nonlinearities, the solutions of these equations may demonstrate irregular, "chaotic" behavior, which is often attributed to the system evolution sensitivity to initial data, also known as the "butterfly effect".

For a biologist, however, models based on classical mathematical techniques such as differential equations, might look too restrictive. Indeed, a typical system consists of a large number of elements (cells), where each cell experiences local nonlinear interactions with its neighbors. It is often argued that *networks* and *automata* can provide better alternative to the traditional approaches being discreet by nature, and defined by specifying the interaction rules between cells. Interestingly, the first automata studies were initiated by mathematicians J. von Neumann and S. Ulam in the early 1950's, inspired by the analogies between the operations of computers and the human brain. At those times the chaotic behavior of differential and difference equations was awaiting its discovery and known mathematical tools were not expressive enough to simulate systems of high level of complexity.[1] Meanwhile, Von Neumann and Ulam also expected that their automata project would help them to come up with new mathematics that would be adequate to describe the patterns, processes, and self-organization of living matter.

Von Neumann's first automation was quite complicated, but 30 years later S. Wolfram discovered remarkably simple rules that may lead to arbitrarily complicated global behavior and visual patterns, as demonstrated in his computer simulations of *elementary cellular automata* (ECA) models [10, 11].

Nowadays, cellular automata is a practical tool to deal with complex systems in many disciplines, competitive even in the fields where traditional techniques are well established (for example, Lattice-Gas method to solve fluid dynamics equations). In some mathematical circles however, these approaches are often considered as an add hoc toolkit rather than a step towards a satisfactory mathematical theory, thus avoiding the complexity as such.

In this paper we would like to emphasize the complementary role of the automata models and abstract mathematical ideas in the studies of complex systems, as originally viewed by Ulam and Von Neumann. We discuss the interrelation between the behavior of patterns and algebraic properties of ECA and their generalizations based on groupoids.

1.2 ECA as groupoids

By definition [11], ECA are discrete dynamical systems that describe the evolution of black and white cells, denoted as 1 and 0 correspondingly, arranged in horizontal lines. Starting from some initial line, where all cells are white and a single cell in the center is black ($\ldots 0001000 \ldots$), for example, the color of the cell on the next line is determined by the color of its three neighbors immediately above it. In order to define a particular ECA rule, it is sufficient to assign "0" or "1" values to 8 triplets, for instance,

$$\begin{matrix} 1\,1\,1 & 1\,1\,0 & 1\,0\,1 & 1\,0\,0 & 0\,1\,1 & 0\,1\,0 & 0\,0\,1 & 0\,0\,0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{matrix}\ . \tag{1.1}$$

It is not difficult to see that there are 256 such rules, which are numbered from 0 to 255 by binary numbers formed from the assigned digits and then converted to decimals. Thus, definition (1.1) corresponds to the rule 110, since $1101110_2 = 110$.

[1] Although Ulam's computer-assisted studies of polynomial iterated maps in the late 1950's demonstrated fascinating limit sets that today would be classified as strange attractors.

It appears, the conventional ECA rules can be easily replaced by the equivalent algebraic objects. Specifically, instead of using 3-cell rules, for example,

$$\begin{matrix} 1\,1\,0 \\ 1 \end{matrix} \quad \begin{matrix} 1\,0\,0 \\ 0 \end{matrix} \;\rightarrow\; \begin{matrix} 1\,1\,0\,0 \\ 1\,0 \end{matrix},$$

one can define "products" of 2-cell blocks, $11 \circ 00 = 10$, etc., *consistent* with the corresponding rule. Denoting $e_1 = 11$, $e_2 = 10$, $e_3 = 01$, $e_4 = 00$, and evaluating similarly all products $e_i \circ e_j,\ i, j = 1..4$, a particular case of ECA can be given by a corresponding multiplication table. Rule 110 (1.1), for instance, has the following table:

$\circ$	e_1	e_2	e_3	e_4
e_1	e_4	e_3	e_1	e_2
e_2	e_1	e_1	e_3	e_4
e_3	e_2	e_1	e_1	e_2
e_4	e_1	e_1	e_3	e_4

This multiplication table defines a four-element *groupoid*, a set G of 4 elements $\{e_i\},\ i = 1..4$ together with a closed binary operation $\circ : G \times G \rightarrow G$, which in the general case is not necessarily associative and/or commutative.

Clearly, each of the ECA rules generates a unique multiplication table, and hence a groupoid. Thus, the ECA evolution can be computed by groupoid multiplications of neighboring elements. Starting from the initial line of 2-block cells $x_1 x_2 ... x_n$, where $x_k \in G,\ k = 1..n$, and evaluating the next $n - 1$ lines results in a single block, denoted by $B_{12...n}$. In general, $n(n-1)/2$ products are required to calculate $B_{12...n}$.

A key question is the following: can we "shortcut" such computations, using less number of operations? It turns out that if groupoid has special properties, the prediction of $B_{12...n}$ may be much more efficient [6, 7]. It can be shown, for example, that the groupoid of rule 90 is a commutative group which requires only $O(n)$ products to calculate the block $B_{12...n}$.

Other ECA based on known and well studied algebraic structures such as semigroups, quasigroups, loops, and groups can also be efficiently predicted. However, the majority of groupoids related to ECA do not belong to any known algebraic structures. For instance, there are only 128 non-equivalent semigroups (associative groupoids) among $4^{16} \approx 4.2 \cdot 10^{10}$ possible 4-element groupoids, first constructed by G. E. Forsythe in 1955 [1].

But what can be said about such general groupoids, which algebraic structure is not known or not studied before? Do they satisfy any identities that would lead to the efficient prediction of the corresponding ECA evolution?

A combinatorial computer-assisted search of identities in these groupoids was previously described in [2], where a list of low order identities for some ECA that show interesting behavior is presented. It appears that groupoids of ECA that demonstrate simple behavior and belong to the class 1 or 2 in Wolfram's classification, exhibit many trivial identities with B-blocks, which provide computational "shortcuts" for efficient prediction. However, much fewer such identities were found in ECA groupoids of class 3 that demonstrate randomness as rule 30. Remarkably, identities with B-blocks were found in groupoids of class 4 ECA for some particular initial conditions. For instance the groupoid of rule 110 satisfies the following identities:

$$B_{2212222} = (x_1x_2)(((x_2x_2^2)x_2)x_2),$$

$$B_{22122222} = B_{21222222} = B_{12222222} = (((x_1x_2)(x_2x_2^2))x_2)x_2^2 = (((x_1x_2)x_2^2)x_2^2)x_2^2 = (x_2(((x_1x_2)x_2^2)x_2))x_2^2 = ((x_1x_2)x_2^2)(x_2^2)^2 = x_2^2(((x_1x_2)x_2^2)x_2^2) = x_2^2(x_2(((x_1x_2)x_2^2)x_2)),$$

$$B_{222122222} = (x_1x_2)(((((x_2x_2^2)x_2)x_2)x_2)x_2) = (x_2((x_1x_2)x_2^2))(x_2^2)^2 = (x_1x_2)((x_2x_2^2)^2x_2) = (x_1x_2)(((x_2((x_2^2x_2)x_2))x_2)x_2) = (x_1x_2)((x_2x_2^2)(x_2(x_2^2x_2))) = (x_1x_2)(((x_2^2x_2)x_2^2)x_2^2) = (x_1x_2)((x_2(x_2^2)^2)x_2^2) = (x_1x_2)((x_2^2x_2)((x_2x_2^2)x_2)) = (x_1x_2)(x_2((x_2(x_2^2)^2)x_2)) = (x_1x_2)(x_2(((x_2^2x_2)x_2^2)x_2)) = (((x_2^2x_2)(x_2(x_1x_2)))x_2)x_2^2.$$

Such identities, however, seem to appear quite randomly, so one may inquire whether there exists an effective description, or a *basis* of the groupoid identities, from which all other identities can be derived. It is well known, however, that groupoids in general are not necessarily *finitely based* (i.e., possess a finite basis of identities). Only 2-element groupoids are finitely based [4], the first example of a three-element nonfinitely based groupoid

$\circ$	e_1	e_2	e_3
e_1	e_1	e_1	e_1
e_2	e_1	e_1	e_2
e_3	e_1	e_3	e_3

(1.2)

was constructed by V. L. Murskii [8]. In particular, he proved that for any $n \geq 3$ the following formula

$$x_1(x_2(x_3 \ldots (x_{n-1}(x_nx_1)\ldots)) = (x_1x_2)(x_n(x_{n-1}\ldots(x_4(x_3x_2))\ldots))$$

is satisfied identically, but cannot be derived from any set of lower degree groupoid (1.2) identities.

Concerning ECA, their equivalence to 4-element groupoids implies that there might exist some set of “shortcut” identities that are impossible to derive from other lower degree identities. We can prove such identities by brute-force exhaustive substitutions, since groupoids are finite structures, but never using the traditional axiomatic approach. The situation resembles Gödel’s incompleteness result, but in the “light form”: brute force is useless in the case of infinite structures.

Clearly, the lack of a finite basis of groupoid identities imposes serious complications on their algorithmic study and explains why one should not be surprised that such simple systems as ECA might be irreducible and show complicated behavior. In fact, there is even no general algorithm to determine whether an arbitrary finite groupoid is finitely based or not, as recently proved by R. McKenzie [5]. Thus, each groupoid/ECA must be studied on the individual basis.

We would like to emphasize especially an important role of nonassociativity as a necessary condition for a nonfiniteness of the groupoid basis and generation of complex patterns in the corresponding automaton. Indeed, one needs at least 6-element semigroup that might be nonfinitely based [9]. Remarkably, Wolfram's experimental studies of cellular automata based on semigroups suggested that a semigroup of at least 6 elements is required to obtain patterns more complicated than nested, regardless of the initial condition [11, p. 887].

1.3 Algebraic cellular automata

So far we were focused on studying ECA as groupoids, but they represent only a subset of 256 out of $4^{16} \approx 4.2 \cdot 10^{10}$ possible 4-element groupoids. It may be surprising that members of such a small subset demonstrate quite a rich behavioral spectrum. Examining the rest of the four billion cases would be time consuming but not impossible for a modern computer.

In light of the above discussion, however, we can introduce cellular automata based on 3- and 2-element groupoids, in analogy with the ECA, and call them further algebraic cellular automata (ACA). For a convenient enumeration we will use digits 0, 1, $\ldots$ instead of e_i notation, because groupoids allow scalar representation. The following 3-element groupoid, for example,

$$\begin{array}{c|ccc} \circ & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 0 & 2 \end{array} \tag{1.3}$$

will be numbered as a decimal 4061, since $12120102_3 = 4061_{10}$, and the number in base 3 is formed by rows of the multiplication table, starting from the top one.

First, as a simplest case, we examine 2-element groupoids. There are only 16 of such groupoids, and we know that they all are finitely based. We would not expect to observe any complicated behavior of the corresponding ACA, but it is interesting to see generic cases of "simple" patterns. To make pictures we use two colors, the element 0 corresponds to the block of 2 black square cells, and 1 to the white one. The most "interesting" patterns obtained from the initial row containing single 0 element and the rest are 1's are shown in Fig. 1.1. Other trivial observed patters are all white, all black, and alternating black and white lines.

We have also done some preliminary studies of 3-element ACA using elements of three colors: 0 is red, 1 is blue, and 2 is white. There are $3^9 = 19683$ different 3-element ACA, and some of them are not finitely based. As an initial condition we used a single red element, the rest were white.

Our experiments showed a wide behavioral spectrum of patterns – from simple nested to random with localized, turbulent structures. One of such patterns is shown in Fig. 1.2.

Clearly, more experiments, especially with different initial conditions, are required to filter the most interesting ACA. This study, however should be accompanied by the investigation of the algebraic structure and identities of corresponding groupoids, which

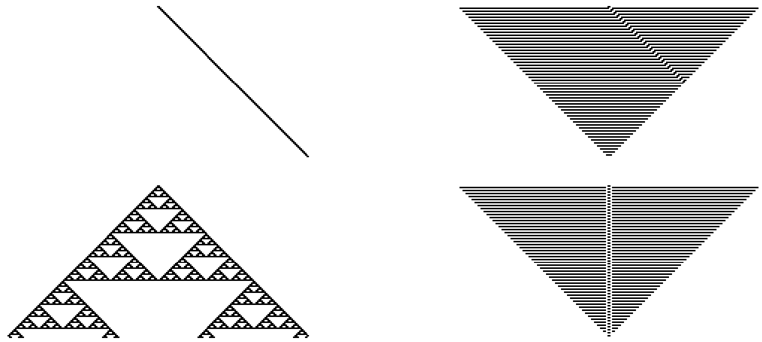

Figure 1.1: Patterns produced by 2-element ACA of number 3 and 9 in the firs column, and 12 and 14 in the second one. First 100 steps of evolution is shown, starting from the single 0 element.

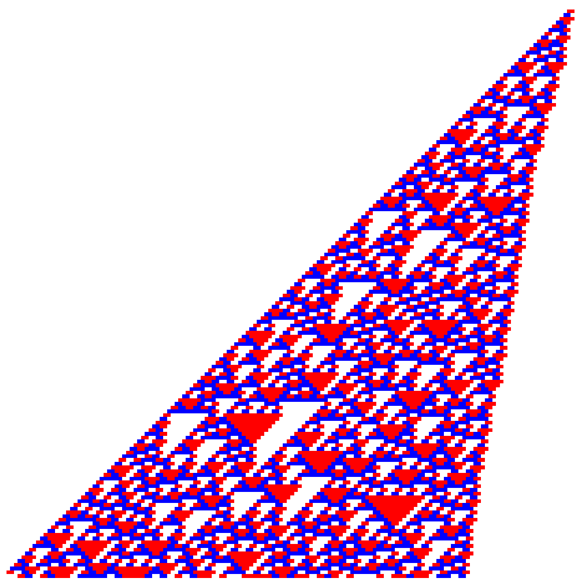

Figure 1.2: First 100 steps evolution of the 3-element ACA number 4061 starting from single red 2-cell block, the rest are white.

might result in the classification of isomorphic families of groupoids. It would also be interesting to compare such families for 3- and 4-element groupoids.

1.4 Discussion

In this paper, we tried to indicate that traditional mathematical methods and structures can be useful in the studies of complex systems, but require some technical and methodological adjustments in order to describe the real world phenomena.

Nonassociative structures, namely groupoids, that allowed equivalent representation of the ECA, are fascinating but unfortunately not very interesting for mathematicians. In particular, Jacobson's classical algebra textbook states [3]:

> One way of trying to create new mathematics form an existing mathematical theory, especially one presented in an axiomatic form, is to generalize the theory by dropping or weakening some of its hypotheses. If we play this axiomatic game with the concept of an associative algebra, we are likely to be led to the concept of a non-associative algebra, which is obtained simply by dropping the associative law of multiplication. If this stage is reached in isolation from other mathematical realities, it is quite certain that one would soon abandon the project, since there is very little of interest that can be said about non-associative algebras in general.

We have seen, however, that axiomatic games are very restrictive, especially when dealing with nonassociative structures. The lifting of the nonassociativity breaks many symmetries, so general groupoids lack some nice structural properties. But such lack of symmetries appears to be an essential property of ECA and many other complex systems.

We have not touched in this paper applications of other nonassociative structures such as nonassociative algebras, which are briefly discussed in [2].

Finally, we would like to stress the increasing importance of modern computers and emerging experimental mathematical techniques, on which we heavily relied in the course of the present study.

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