Mathematical model of conflict and cooperation with non-annihilating multi-opponent

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We introduce first our multi-opponent conflict model and consider the associated dynamical system for a finite collection of positions. Opponents have no strategic priority with respect to each other. The conflict interaction among the opponents only produces a certain redistribution of common area of interests. The limiting distribution of the conflicting areas, as a result of 'infinite conflict interaction for existence space, is investigated. Next we extend our conflict model and propose conflict and cooperation model, where some opponents cooperate with each other in the conflict interaction. Here we investigate the evolution of the redistribution of the probabilities with respect to the conflict and cooperation composition, and determine invariant states by using computer experiment.

1.1 Introduction

Decades of research on social conflict has contributed to our understanding of a variety of key social, and community-based aspects of conflict escalation. However, the field has yet to put forth a formal theoretical model that links these components to the basic underlying mechanisms. This paper presents such models: dynamical-systems model of conflict and cooperation. We propose that it is particularly useful to conceptualize ongoing.

In biology and social science, conflict theory states that the society or organization functions in a way that each individual participant and its groups struggle to maximize their benefits, which inevitably contributes to social change such as changes in politics and revolutions. This struggle generates conflict interaction. Usually conflict interaction takes place in micro level i.e in individual interaction or in semi-macro level i.e. in group interaction. Then these interactions give impact on macro level. Here we would like to highlight the relation between macro level phenomena and semi macro level dynamics. We construct a framework of conflict and cooperation model by using group dynamics.

First we introduce a conflict composition for multi-opponent and consider the associated dynamical system for a finite collection of positions. Opponents have no strategic priority with respect to each other. The conflict interaction among the opponents only produces a certain redistribution of common area of interests. We have developed this model based on some recent papers by V. Koshmanenko, which describes a conflict model for non-annihilating two opponents. By means of conflict among races how segregation emerges in the society is shown. Next we extend our conflict model to conflict and cooperation model, where some opponents cooperate with each other in the conflict interaction. Here we investigate the evolution of the redistribution of the probabilities with respect to the conflict and cooperation composition, and determine invariant states.

1.2 Mathematical model of conflict with multi-opponent

In some recent papers V. Koshmanenko (2003, 2004) describes a conflict model, for non-annihilating *two* opponent groups through their group dynamics. But we observe that there are many multi-opponent situations, in our social phenomena, where they are making conflicts to each other. For example, there are multi race (e.g., Black, White, Chinese, Hispanic, etc), multi religion (e.g., Islam, Christian, Hindu, etc) and different political opinions exist in the society and because of their differences they have conflicts to each other. Therefore it is very important to construct conflict model for multi-opponent situation to understand realistic conflict situations in the society.

In order to give a good understanding of our model to the reader, we firstly explain it for the case of four opponents denoted by $A_1, A_2,\ A_3$and A_4 and four positions. We denote by $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ the set of positions which A_1, A_2, A_3 and A_4 try to occupy. Hence $\omega_1, \omega_2,\ \omega_3$ and ω_4 represents different positions in Ω. By a social scientific interpretation, each $\omega_j, j = 1, 2, 3, 4$ represent an area of a big city Ω. Let μ_0, ν_0, γ_0 and η_0 denote the probability measures on Ω. We define the probability that the opponents $A_1, A_2,\ A_3$and A_4 occupy the position $\omega_j, j = 1, 2, 3, 4$ with probabilities $\mu_0(\omega_j), \nu_0(\omega_j),\ \gamma_0(\omega_j)$ and $\eta_0(\omega_j)$ respectively. As we are thinking about the probability measures and a priori the opponents are assumed to be non-annihilating, it holds that

$$\sum_{j=1}^{4} \mu_0(\omega_j) = 1, \sum_{j=1}^{4} \nu_0(\omega_j) = 1, \sum_{j=1}^{4} \gamma_0(\omega_j) = 1, \sum_{j=1}^{4} \eta_0(\omega_j) = 1. \tag{1.1}$$

Since $A_1, A_2,\ A_3$and A_4 are incompatible, this generates a conflicting interaction and we express this mathematically in a form of conflict composition. Namely, we define the conflict composition in terms of the conditional probability to occupy, for example, ω_1 by each of the opponents. Therefore for the opponent A_1 this conditional probability should be proportional to the product,

$$\mu_0(\{\omega_1\}) \times \nu_0(\{\omega_2\}, \{\omega_3\}, \{\omega_4\}) \times \gamma_0(\{\omega_2\}, \{\omega_3\}, \{\omega_4\}) \times \eta_0(\{\omega_2\}, \{\omega_3\}, \{\omega_4\}). \tag{1.2}$$

We note that this corresponds to the probability for A_1 to occupy ω_1 and the probability for A_2, A_3 and A_4 to be absent in that position ω_1. Similarly for the opponents A_2, A_3 and A_4 we define the corresponding quantities. As a result, we obtain a re-distribution of the conflicting areas. We can repeat the above described procedure for infinite number of times, which generates a trajectory of the conflicting dynamical system. The limiting distribution of the conflicting areas is investigated. The essence of the conflict is that the opponents A_1, A_2, A_3 and A_4 can not simultaneously occupy a questionable position ω_j. Given the initial probability distribution:

$$\begin{pmatrix} p_{11}^{(0)} & p_{12}^{(0)} & p_{13}^{(0)} & p_{14}^{(0)} \\ p_{21}^{(0)} & p_{22}^{(0)} & p_{23}^{(0)} & p_{24}^{(0)} \\ p_{31}^{(0)} & p_{32}^{(0)} & p_{33}^{(0)} & p_{34}^{(0)} \\ p_{41}^{(0)} & p_{42}^{(0)} & p_{43}^{(0)} & p_{44}^{(0)} \end{pmatrix} \tag{1.3}$$

the conflict interaction for each opponent for each position is defined as follows:

$$\begin{gathered} p_{11}^{(1)} := \tfrac{1}{z_1^{(0)}} p_{11}^{(0)}(1-p_{21}^{(0)})(1-p_{31}^{(0)})(1-p_{41}^{(0)}); p_{12}^{(1)} := \tfrac{1}{z_1^{(0)}} p_{12}^{(0)}(1-p_{22}^{(0)})(1-p_{32}^{(0)})(1-p_{42}^{(0)}); \\ p_{13}^{(1)} := \tfrac{1}{z_1^{(0)}} p_{13}^{(0)}(1-p_{23}^{(0)})(1-p_{33}^{(0)})(1-p_{43}^{(0)}); p_{14}^{(1)} := \tfrac{1}{z_1^{(0)}} p_{14}^{(0)}(1-p_{24}^{(0)})(1-p_{34}^{(0)})(1-p_{44}^{(0)}); \\ \text{and so on.} \end{gathered} \tag{1.4}$$

where the normalizing coefficient

$$z_1^{(0)} = \sum_{j=1}^{3} p_{1j}^{(0)}(1-p_{2j}^{(0)})(1-p_{3j}^{(0)})(1-p_{4j}^{(0)}). \tag{1.5}$$

Thus after one conflict the probability distributions changes in the following way:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{array} \\ \begin{pmatrix} p_{11}^{(0)} & p_{12}^{(0)} & p_{13}^{(0)} & p_{14}^{(0)} \\ p_{21}^{(0)} & p_{22}^{(0)} & p_{23}^{(0)} & p_{24}^{(0)} \\ p_{31}^{(0)} & p_{32}^{(0)} & p_{33}^{(0)} & p_{34}^{(0)} \\ p_{41}^{(0)} & p_{42}^{(0)} & p_{43}^{(0)} & p_{44}^{(0)} \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{array} \\ \begin{pmatrix} p_{11}^{(1)} & p_{12}^{(1)} & p_{13}^{(1)} & p_{14}^{(1)} \\ p_{21}^{(1)} & p_{22}^{(1)} & p_{23}^{(1)} & p_{24}^{(1)} \\ p_{31}^{(1)} & p_{32}^{(1)} & p_{33}^{(1)} & p_{34}^{(1)} \\ p_{41}^{(1)} & p_{42}^{(1)} & p_{43}^{(1)} & p_{44}^{(1)} \end{pmatrix} \end{array} \tag{1.6}$$

Thus by induction after kth conflict the probability distributions changes in the following way:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{array} \\ \begin{pmatrix} p_{11}^{(k-1)} & p_{12}^{(k-1)} & p_{13}^{(k-1)} & p_{14}^{(k-1)} \\ p_{21}^{(k-1)} & p_{22}^{(k-1)} & p_{23}^{(k-1)} & p_{24}^{(k-1)} \\ p_{31}^{(k-1)} & p_{32}^{(k-1)} & p_{33}^{(k-1)} & p_{34}^{(k-1)} \\ p_{41}^{(k-1)} & p_{42}^{(k-1)} & p_{43}^{(k-1)} & p_{44}^{(k-1)} \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{array} \\ \begin{pmatrix} p_{11}^{(k)} & p_{12}^{(k)} & p_{13}^{(k)} & p_{14}^{(k)} \\ p_{21}^{(k)} & p_{22}^{(k)} & p_{23}^{(k)} & p_{24}^{(k)} \\ p_{31}^{(k)} & p_{32}^{(k)} & p_{33}^{(k)} & p_{34}^{(k)} \\ p_{41}^{(k)} & p_{42}^{(k)} & p_{43}^{(k)} & p_{44}^{(k)} \end{pmatrix} \end{array} \tag{1.7}$$

The general formulation of this model for multi opponents and multi positions and its theorem for limiting distribution is given in our recent paper Salam, Takahashi (2006). We also investigated this model by using empirical data but because of page restriction we can not include that in this paper.

1.2.1 Computer Experimental Results

In our simulation results $M^{(0)}$ is the initial matrix where row vectors represent the distribution of each races. There are four races white, black, Asian and Hispanic denoted by A_1, A_2, A_3 and A_4 respectively. $\omega_1, \omega_2, ...,$ represents the districts of a city. Here all three races moving to occupy these districts, thus the conflict appear. Here $M^{(\infty)}$ gives the convergent or equilibrium matrix. There are several graphs in each figure. Each graph shows the trajectory correspond to the each element of the matrix. In each graph x-axis represent the number of conflict and y-axis represent the probability to occupy that position.

In result 1, which is given below, we observe that opponent A_1 has biggest probability in city ω_1 and after 9 interaction it occupy this city. Opponent A_2 has bigger probability to occupy city ω_2 and ω_4. But in city ω_4 opponent A_4 has the biggest probability to occupy since the opponents are non-annihilating opponent A_2 gather in ω_2 and occupy this city after 9 conflict interactions and opponent A_4 occupy the city ω_4 after 9 conflict interaction. Opponent A_3 also has bigger probability to occupy city ω_2 and ω_3. As opponents are non-annihilating and A_2 occupy ω_2, opponent A_3 occupy ω_3 after 9 conflict interaction. Thus each races segregated into each of the cities. This result shows how segregation appear due to conflict.

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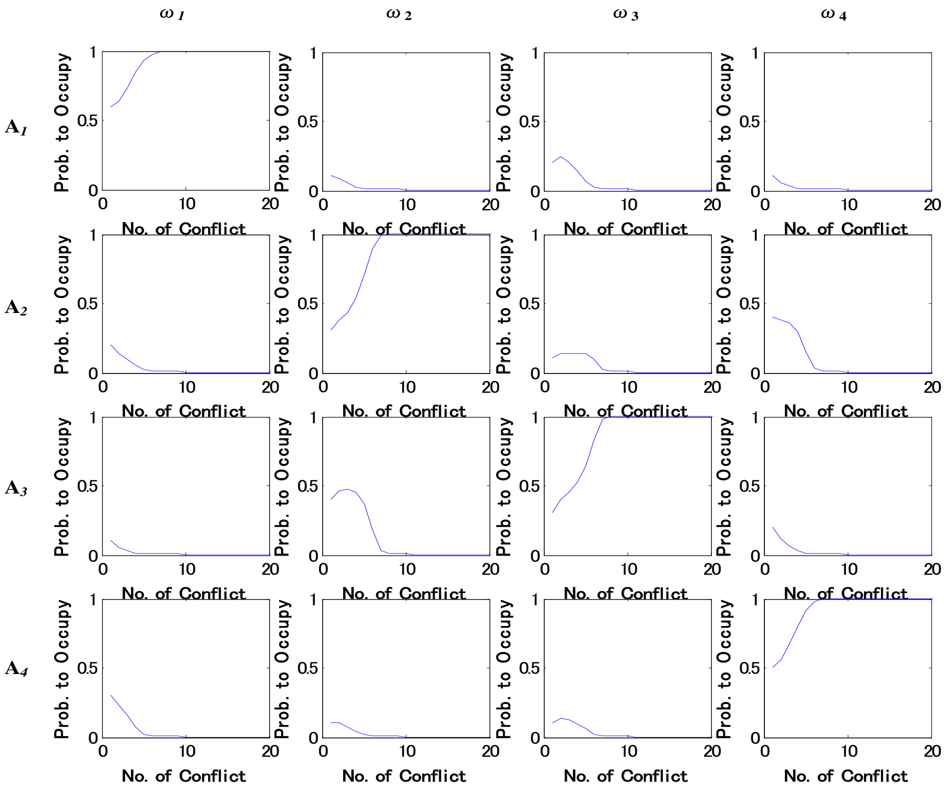
$$M^{(0)} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{array} \\ \begin{pmatrix} 0.6 & 0.1 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.1 & 0.4 \\ 0.1 & 0.4 & 0.3 & 0.2 \\ 0.3 & 0.1 & 0.1 & 0.5 \end{pmatrix} \end{array} \qquad M^{(\infty)} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$



1.3 Mathematical Model of Conflict and Cooperation

Suppose that A_1 and A_2 cooperate with each other in this conflict interaction. We express this mathematically in a form of conflict and cooperation composition. Namely, we define the conflict and cooperation composition in terms of the conditional probability to occupy, for example, ω_1 by each of the opponents. Therefore for the opponent A_1 and A_2 this conditional probability should be proportional to the product,

$$[\mu_0(\{\omega_1\})+\nu_0(\{\omega_1\})-\mu_0(\{\omega_1\}\times\nu_0(\{\omega_1\})]\times\gamma_0(\{\omega_2\},\{\omega_3\},\{\omega_4\})\times\eta_0(\{\omega_2\},\{\omega_3\},\{\omega_4\}). \tag{1.8}$$

We note that this corresponds to the probability for A_1 and A_2 to occupy ω_1 and the probability for A_3 and A_4 to be absent in that position ω_1.

For the opponent A_3 this conditional probability should be proportional to the product,

$$\gamma_0(\{\omega_1\}) \times \mu_0(\{\omega_2\},\{\omega_3\},\{\omega_4\}) \times \nu_0(\{\omega_2\},\{\omega_3\},\{\omega_4\}) \times \eta_0(\{\omega_2\},\{\omega_3\},\{\omega_4\}). \tag{1.9}$$

Similarly for the opponent A_4 we define the corresponding quantities. As a result, we obtain a re-distribution of the conflicting areas. We can repeat the above described procedure for infinite number of times, which generates a trajectory of the conflict and cooperation dynamical system. The limiting distribution of the conflicting areas is investigated by using computer experiment. Given the initial probability distribution (1.3) the conflict and cooperation composition for each opponent for each position is defined as follows:

$$\begin{aligned}
&p_{11}^{(1)} := \tfrac{1}{z_1^{(0)}}(p_{11}^{(0)} + p_{21}^{(0)} - p_{11}^{(0)}p_{21}^{(0)})(1-p_{31}^{(0)})(1-p_{41}^{(0)}) = p_{21}^{(1)};\\
&p_{31}^{(1)} := \tfrac{1}{z_3^{(0)}}p_{31}^{(0)}(1-p_{11}^{(0)})(1-p_{21}^{(0)})(1-p_{41}^{(0)}); p_{41}^{(1)} := \tfrac{1}{z_4^{(0)}}p_{41}^{(0)}(1-p_{11}^{(0)})(1-p_{21}^{(0)})(1-p_{31}^{(0)});\\
&\text{and so on, } z_i^{(0)}\text{'s are the normalizing coefficients.}
\end{aligned} \tag{1.10}$$

Thus after one conflict the probability distributions changes as (1.6), but the quantities are different from the previous model and by induction after kth conflict the probability distributions changes as as (1.7), but the quantities are also different.

1.3.1 Computer Experimental Results

In this computer experimental result opponent A_1 and A_2 cooperate each other. We observe that in position ω_1 opponent A_1 has biggest probability to occupy this position. As opponent A_1 and A_2 cooperate each other, both of them occupy this position after 23 interactions. In position ω_3 opponent A_3 has the biggest probability to occupy but as A_1 and A_2 cooperate each other they occupy this position after 23 interactions. Since the opponents are non-annihilating opponent A_3 and A_4 occupy the positions ω_2 and ω_4 respectively.

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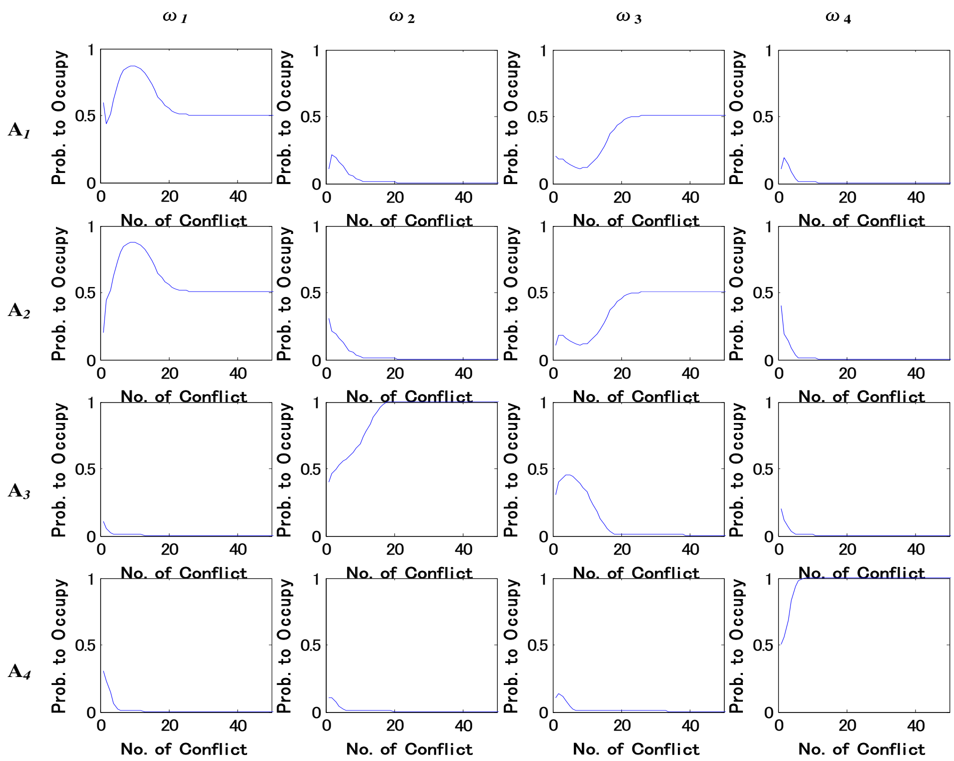
$$M^{(0)} = \begin{array}{c c} & \begin{array}{cccc} \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0.6 & 0.1 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.1 & 0.4 \\ 0.1 & 0.4 & 0.3 & 0.2 \\ 0.3 & 0.1 & 0.1 & 0.5 \end{pmatrix} \end{array} \qquad M^{(\infty)} = \begin{array}{c c} & \begin{array}{cccc} \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$



1.4 Conclusion

Since social, biological, and environmental problems are often extremely complex, involving many hard-to-pinpoint variables, interacting in hard-to-pinpoint ways. Often it is necessary to make rather severe simplifying assumptions in order to be able to handle such problems, our model can be refine by including more parameters to include more broad conflict situations. Our conflict model did not have destructive effects. One way to alter this assumption is to make the population mortality rate grow with conflict efforts. We suspect these changes would dampen the dynamics. We observed that for multi-opponent conflict model each opponent can occupy only one position but because of cooperation two opponents who cooperate each other can occupy two positions with same initial distribution.

We emphasize that our framework differ from traditional game theoretical approach. Game theory makes use of the payoff matrix reflecting the assumption that the set of outcomes is known. The Nash equilibrium, the main solution concept in analytical game theory, cannot make precise predictions about the outcome of repeated games. Nor can it tell us much about the dynamics by which a population of players moves from one equilibrium to another. These limitations have motivated us to use stochastic dynamics in our conflict model. Our framework also differ from Schelling's segregation model in several respects. Specially Schelling's results are derived from an extremely small population and his model is limited to only two race-ethnic groups. Unlike Schelling's model we do not suppose the individuals' choices here we consider group's choice.

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