

# Epsilon-Pseudo-Orbits and Applications

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## 1. Introduction

We consider dynamical systems consisting of the iteration of continuous functions on compact metric spaces. Basic definitions and results on chain recurrence and the Conley Decomposition Theorem in this setting are presented. An  $\varepsilon$ -pseudo-orbit approximation for the dynamics, with  $\varepsilon$  of fixed size, is presented as a mathematical representation of a computer model of such a discrete dynamical system. We see that the Conley decomposition of a space can be approximated by an  $\varepsilon$ -coarse Conley decomposition in this setting.

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## 2. The Setting

Charles Conley presents in [1] a fundamental result now called the Conley Decomposition Theorem. It provides for the decomposition of every flow on a compact metric space into a part that exhibits a particular type of recurrence and a part in which the dynamics are essentially one-way. The theorem and its implications have been investigated in several settings, from semi-flows to homeomorphisms; see, for example, [2-7]. Here, we consider a model for corresponding results in one of the standard contemporary introductions to the study of dynamical systems: the iteration of continuous functions on computers.

Let  $(X, d)$  denote a compact metric space and  $f : X \rightarrow X$  denote a continuous function mapping  $X$  into itself. (Notice that  $f$  is not necessarily invertible.) The *forward orbit* of  $x_0 \in X$  is the set of all forward iterates of  $x_0$ . The point  $x_0$  is a *fixed point* for  $f$  if  $f(x_0) = x_0$ . The point  $x_0$  is a *periodic point* for  $f$  if  $f^n(x_0) = x_0$  for some  $n > 0$ , and its forward orbit is called a *periodic forward orbit*. If  $x_n = x_0$  but  $x_k \neq x_0$  for  $0 < k < n$ , then  $n$  is called the *period* of the forward orbit. Finally, an *orbit* is a bi-infinite sequence of points  $(\dots, x_{-2}, x_{-1}, x_0, x_1, \dots)$  such that  $f(x_n) = x_{n+1}$  for every integer  $n$ .

### 3. Basic Definitions and Results

To represent the output of a computer performing the iteration of the function, we consider the idea of  $\varepsilon$ -pseudo-orbit. This idea goes back to Anosov, Bowen, and Conley in [8-11, 1]. A sequence of points  $(x_0, x_1, \dots, x_n)$  in  $X$  which satisfies the condition  $d(f(x_{k-1}), x_k) < \varepsilon$  for  $k = 1, 2, \dots, n$  is called an  $\varepsilon$ -pseudo-orbit of length  $n$ .

For  $x$  and  $y$  in  $X$ , we will write  $x > y$  to mean that for every  $\varepsilon > 0$ , there is an  $\varepsilon$ -pseudo-orbit from  $x$  to  $y$ ; that is, for every  $\varepsilon > 0$  there is an  $n > 0$  so that  $(x_0, x_1, \dots, x_n)$  is an  $\varepsilon$ -pseudo-orbit with  $x_0 = x$  and  $x_n = y$ . Notice that  $x > y$  does not simply mean that there is an  $\varepsilon$ -pseudo-orbit from  $x$  to  $y$ ; it is a statement for *all*  $\varepsilon > 0$ .

A point  $x \in X$  is called *chain recurrent* if  $x > x$ . That is,  $x$  is chain recurrent if there is an  $\varepsilon$ -pseudo-orbit from  $x$  back to itself for any choice of  $\varepsilon > 0$ .  $R(f) =$  the *chain recurrent set of  $f$*   $= \{x \in X : x \text{ is chain recurrent}\}$ . Since any true orbit is an  $\varepsilon$ -pseudo-orbit for any  $\varepsilon$ , any periodic point is chain recurrent. Some of the basic results on chain recurrence can be found in [1-3, 5-6].

For  $x$  and  $y$  in  $X$ , we will write  $x \sim y$  to mean that  $x > y$  and  $y > x$ . That is, for every  $\varepsilon > 0$  there exist  $\varepsilon$ -pseudo-orbits  $(x_0, x_1, \dots, x_m)$  and  $(y_0, y_1, \dots, y_n)$  with  $x_0 = x = y_n$  and  $x_m = y = y_0$ . It can be shown that  $R(f)$  is closed and that the relation " $\sim$ " is an equivalence relation on  $R(f)$ . These and other results are proved in this setting in [6].

The relation " $\sim$ " partitions  $R(f)$  into equivalence classes, sometimes called *chain components* or *chain classes*. Each chain class is *chain transitive*: that is, within each chain class, any two points have an  $\varepsilon$ -pseudo-orbit connecting them, for any  $\varepsilon > 0$ . Given any two points within a chain class and any  $\varepsilon > 0$ , there is a periodic  $\varepsilon$ -pseudo-orbit containing both points. Chain classes are by definition the maximal chain transitive sets in the sense that no chain class lies within a chain transitive set that is strictly larger than itself.

A key result in [1, 4-5], proved in this setting in [6], is that the chain recurrent set can be characterized in terms of the attractors of the space. If  $A^*$  is the repeller complementary to attractor  $A$ , we have the following.

**Theorem.**  $R(f) = \bigcap \{ A \cup A^* : A \text{ is an attractor for } f \text{ on } X \}$ .

Chain classes of the chain recurrent set are found in intersections of attractors and repellers of the system. A point not in the chain recurrent set lies in the domain(s) of attraction for some attractor(s) and under the action of the function heads toward its  $\omega$ -limit set in a unique chain class. Such points outside the chain recurrent set exhibit behavior that is said to be gradient-like.

The idea of a gradient-like portion of the space is an extension from gradient flows of the idea of functions that decrease on solutions, called *Lyapunov functions*. A space is called *gradient-like* if there is some continuous real-valued function that is strictly decreasing on nonconstant solutions.

**Definition.** A *complete Lyapunov function* for the space  $X$  with respect to a continuous function  $f$  is a continuous, real-valued function  $g$  on  $X$  satisfying:

- (a)  $g(f(x)) < g(x)$  for  $x \in R(f)$ ;
- (b)  $g(R(f))$  is a compact nowhere dense subset of  $\mathbf{R}$ ;
- (c) if  $x, y \in R(f)$ , then  $g(x) = g(y)$  if and only if  $x \sim y$ ; that is, for any  $c \in g(R(f))$ ,  $g^{-1}(c)$  is a chain class of  $R(f)$ .

**Theorem.** *If  $f$  is a continuous function on a compact metric space  $X$ , then there is a complete Lyapunov function  $g: X \rightarrow \mathbf{R}$  for  $f$ .*

The structure and the Lyapunov function combine to characterize the basic dynamical composition of the space in what is sometimes called the Fundamental Theorem of Dynamical Systems. See [12].

**Theorem: The Conley Decomposition Theorem.** *Let  $(X, d)$  denote a compact metric space and  $f: X \rightarrow X$  denote a continuous function mapping  $X$  into itself. Then the dynamical system consisting of the iteration of  $f$  on  $X$  decomposes the space  $X$  into a chain recurrent part and a gradient-like part.*

The partial order “ $>$ ” on the points of the space  $X$  generates a partial order on the chain classes of  $X$  which is then reflected by the complete Lyapunov function on  $X$ . If  $\{C_k\}$  represents the chain classes of  $R(f)$ , then if there is an orbit from  $C_i$  to  $C_j$ , then  $g(C_i) > g(C_j)$ , and the components of  $R(f)$  can be ordered respecting the decreasing requirement of  $g$ . In general, there are many orderings of the components of  $R(f)$  by different complete Lyapunov functions, all of which respect the order imposed by the dynamics. Finally, it is the chain classes and appropriate unions with connecting orbits from the gradient-like portion of the space which form the isolated invariant sets of the space. See [1] for the original theory in the setting of flows and [6] for the results in this setting.

#### 4. New Definitions and Results

Computer models of dynamical systems do not represent the more theoretical “for every  $\varepsilon > 0$ ” but rather reflect a fixed finite bound on the deviation size at each iteration. The focus of the ongoing research described here is to fix  $\varepsilon > 0$ , make the definitions that correspond to the standard definitions that let  $\varepsilon$  go to zero, and compare results to those in the standard setting. For proofs and related results, see [7]. For  $x$  and  $y$  in  $X$ , we will write  $x >^\varepsilon y$  to mean that for the fixed  $\varepsilon > 0$ , there is an  $\varepsilon$ -pseudo-orbit from  $x$  to  $y$ . That is, there is an  $n > 0$  so that  $(x_0, x_1, \dots, x_n)$  is an  $\varepsilon$ -pseudo-orbit with  $x_0 = x$  and  $x_n = y$ . There is no requirement beyond the existence of one  $\varepsilon$ -pseudo-orbit for the given  $\varepsilon > 0$ . Then a point  $x \in X$  is called  $\varepsilon$ -chain recurrent if  $x >^\varepsilon x$ . That is,  $x$  is  $\varepsilon$ -chain recurrent if there is an  $\varepsilon$ -pseudo-orbit from  $x$  back to itself for our given fixed  $\varepsilon > 0$ . We define the  $\varepsilon$ -chain recurrent set of  $f$  to be  $R_\varepsilon(f) = \{x \in X : x \text{ is } \varepsilon\text{-chain recurrent}\}$ . We have the following results.

**Proposition.** For any  $\varepsilon > 0$ ,  $R(f) \subseteq R_\varepsilon(f)$ .

**Proposition.**  $R(f) = \bigcap R_\varepsilon(f)$ .

For  $x$  and  $y$  in  $X$ , we will write  $x \sim^\varepsilon y$  to mean that  $x >^\varepsilon y$  and  $y >^\varepsilon x$ . That is, for our fixed  $\varepsilon$ , there exist  $\varepsilon$ -pseudo-orbits  $(x_0, x_1, \dots, x_m)$  and  $(y_0, y_1, \dots, y_n)$  with  $x_0 = x = y_n$  and  $x_m = y = y_0$ . Then “ $\sim^\varepsilon$ ” is defined on the  $\varepsilon$ -chain recurrent set and partitions  $R_\varepsilon(f)$  into equivalence classes, so that the relation “ $\sim^\varepsilon$ ” is an equivalence relation on  $R_\varepsilon(f)$ .

The Conley Decomposition Theorem has an analogue in the case of fixed  $\varepsilon$ . See [7].

**Theorem: The Conley Decomposition Theorem for Fixed  $\varepsilon$ .** Let  $(X, d)$  denote a compact metric space and  $f : X \rightarrow X$  denote a continuous function mapping  $X$  into itself. Let  $\varepsilon > 0$  be fixed. Then the dynamical system consisting of the iteration of  $f$  on  $X$  decomposes the space  $X$  into an  $\varepsilon$ -chain recurrent part and a gradient-like part. That is, the one-way nature of the orbits of points outside the  $\varepsilon$ -chain recurrent can be described by that part of the Lyapunov function inherited from the chain recurrent decomposition which decreases on those orbits.

The main result is the following.

**Theorem.**  $R_\varepsilon(f) \rightarrow R(f)$  as  $\varepsilon \rightarrow 0+$ , where the set convergence is with respect to the Hausdorff metric.

The collection of  $\varepsilon$ -chain recurrent sets converges in the Hausdorff metric to the chain recurrent set of the system. That is, as we take  $\varepsilon$  smaller and smaller, the  $\varepsilon$ -chain recurrent set is a better and better approximation of the chain recurrent set in a very well-defined way. This result suggests that the  $\varepsilon$ -chain recurrent set offers an appropriate representation of the chain recurrent set if  $\varepsilon$  is the level of greatest possible accuracy.

For example, in computer representations of dynamical systems of iterated maps, smaller and smaller values of  $\varepsilon$  give closer and closer views of the chain recurrent set as seen through the  $\varepsilon$ -chain recurrent sets. We can study the  $\varepsilon$ -decomposition of the space and consider an  $\varepsilon$ -graph of the decomposition corresponding to the decomposition graph of the transitivity relation as described in the standard setting; see [7], for example.

Of course, there are phenomena that one could expect to miss in computer experiments for any finite value of  $\varepsilon$ . On the other hand, it is possible for  $R_\varepsilon(f)$  to have infinitely-many chain classes; see [13] for an example. With either a finite or infinite number of chain classes,  $R_\varepsilon(f)$  both shows some of the significant dynamics and fails to capture all the dynamical structure that the standard chain recurrent set reflects. However, if the chain recurrent set has only finitely many chain classes, then by the convergence in the Hausdorff metric, there is an  $\varepsilon > 0$  so that  $R_\varepsilon(f)$  has the same number of classes.

**Proposition.** *If the number of chain classes in the chain recurrent set is finite, then the  $\varepsilon$ -graph of the system and the standard decomposition graph of the system are the same for some  $\varepsilon > 0$ .*

The  $\varepsilon$ -decomposition of the space, then, provides a good tool for the analysis of the dynamics on the space. If the actual decomposition of the space has finitely many chain classes, then for a sufficiently small  $\varepsilon$ , the  $\varepsilon$ -decomposition reflects the dynamics of the space. If the actual decomposition of the space has infinitely many chain classes, then smaller and smaller values of  $\varepsilon$  provide increasingly accurate reflections of the dynamics.

## 5. Applications

One recent application of  $\varepsilon$ -pseudo-orbits is in the field of neurosciences. A spike in action potential reflecting neural excitability can be described as a subthreshold oscillation having a large amplitude periodic pseudo-orbit nearby. See [14] for details.

Recent applications of  $\varepsilon$ -pseudo-orbits within mathematics are varied, from shadowing and approximation to the structure matrix, attractors, and computation. See [15-20] for some current uses of  $\varepsilon$ -pseudo-orbits across a spectrum of dynamical systems contexts.

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